Working papers series

WP ECON 11.12

A common ground for resource and welfare egalitarianism

Juan D. Moreno-Ternero (U. Pablo de Olavide and CORE)

John E. Roemer (Yale University)

JEL Classification numbers: D63

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Department of Economics

A common ground for resource and welfare egalitarianism*

Juan D. Moreno-Ternero† **John E. Roemer**‡

December 13, 2011

Abstract

Resource egalitarianism and welfare egalitarianism are two focal conceptions of distributive justice. We show in this paper that they share a solid common ground. To do so, we analyze a simple model of resource allocation in which agents' abilities (to transform the resource into an interpersonally comparable outcome) and starting points may differ. Both conceptions of egalitarianism are naturally modeled in this context as two allocation rules. The two rules are jointly characterized by the combination of three appealing axioms: priority, solidarity, and composition.

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*An earlier version of this article circulated as CORE Discussion Paper under the title "Axiomatic resource allocation for heterogeneous agents". We thank Marc Fleurbaey, François Maniquet, Hervé Moulin, an Advisory Editor of this journal and two anonymous referees for helpful comments and suggestions. We also thank the audiences at the Third Workshop on Social Decisions (Málaga), the Ninth International Meeting of the Society for Social Choice and Welfare (Montreal) and the Workshop on Fairness, Equality of Opportunity and Public Economics (Louvain-la-Neuve), as well as at seminars at the University of Copenhagen and Seoul National University. Moreno-Ternero acknowledges financial support from the Spanish Ministry of Science and Innovation (ECO2008-03883, ECO2011-22919) as well as from the Andalusian Department of Economy, Innovation and Science (SEJ-4154, SEJ-5980) via the "FEDER operational program for Andalusia, 2007-2013".

†Department of Economics, Universidad Pablo de Olavide and CORE, Université catholique de Louvain.

‡Elizabeth S. and A. Varick Stout Professor of Political Science and Economics, Yale University.

1 Introduction

A significant amount of effort in political philosophy over the past few decades has been concentrated on the issue of distributive justice. A central impetus for this should be attributed to John Rawls's theory (e.g., Rawls, 1971), which constituted an influential endorsement for egalitarianism. Egalitarian doctrines tend to express the idea that all human persons are equal in fundamental worth or moral status. In spite of this seemingly unquestionable idea, egalitarianism is a contested concept in social and political thought. Discussions in moral philosophy have offered us a wide menu in answer to the question: equality of what? (e.g., Sen, 1980). In other words, if one is an egalitarian, what should one wish to equalize? Two well-known (and focal) theories have been singled out to answer this question in distributive justice. *Resource egalitarianism* holds that a distributional scheme treats people as equals when it distributes or transfers so that no further transfer would leave their shares of the total resources more equal (e.g., Dworkin, 1981b). *Welfare egalitarianism* holds that it treats them as equals when it distributes or transfers so that no further transfer would leave them more equal in welfare (e.g., Dworkin, 1981a). It is plain that both theories will offer different advice in many concrete cases. Nevertheless, we show in this paper that they have a solid common ground, as we shall derive them as the only allocation methods satisfying several appealing principles.

More precisely, imagine the following basic problem. There is an amount of *wealth* to be allocated among individuals, each of whom possesses a capability to transform wealth into some given valued (but non-transferable) outcome, and the achievements of individuals, with regard to that outcome, are interpersonally comparable. Think, for instance, of life expectancy as a function of investment in health care. In resource allocation problems of this sort, if individuals have equal rights over resources, resource egalitarianism (i.e., to distribute the available resource equally among all agents) and welfare (or outcome) egalitarianism (i.e., to distribute the resource among the population so as to equalize, as much as possible, the outcomes among them) are usually two focal points of distribution. We show that in this context both rules can actually be characterized together by combining three axioms. More precisely, we will consider the so-called *priority* axiom (e.g., Moreno-Ternero and Roemer, 2006), which imposes a positive discrimination (but only to a certain extent) towards the less capable of transforming resource into outcome; the so-called *solidarity* axiom (e.g., Thomson, 1983a,b; Roemer, 1986) formalizing the idea that changes on the number of individuals and the available wealth should affect all incumbent agents in the same direction; and the so-called *composition* axiom (e.g., Young, 1988; Moulin, 2000) pertaining to the behavior of a rule with respect to tentative allocations based on an incorrect estimation of the available wealth. Our main result will say that the two rules described above are the only ones satisfying these three axioms.

The rest of the paper is organized as follows. Section 2 presents the model of resource allocation upon which we will base our analysis. Section 3 is devoted to the results. Section 4 summarizes the connections of our work with some related literature. Section 5 concludes. For a smooth passage, we defer some proofs and provide them in the appendix.

2 The model

Let $\mathbb{I}$ represent a population of agents (a set with an infinite number of members) who transform a resource, sometimes called wealth, into an objectively measurable (and interpersonally comparable) outcome, sometimes called welfare. For each $i \in \mathbb{I}$, let $u_i : \mathbb{R}_+ \to \mathbb{R}_+$ be the individual function that models this process.[1] We assume that, for each i, u_i is continuous, strictly increasing and unbounded. We also assume that $\mathcal{U} = \{u_i : i \in \mathbb{I}\}$ constitutes a sufficiently *rich* domain. More precisely, we assume that $\mathcal{U}$ contains all positive (and increasing) piece-wise linear functions and, for reasons that will become clear later in the text, that it is closed under horizontal translations. Formally, if there exist sequences $\{\alpha_j, \beta_j, \lambda_j\}_{j=1}^{k}$ such that,

- For each $j = 1, \ldots, k$, $\alpha_j \in \mathbb{R}_{++}$, $\lambda_j \in \mathbb{R}_+$ and $\beta_j \in \mathbb{R}$;
- $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$;
- For each $j = 1, \ldots, k-1$, $\alpha_j \lambda_{j+1} + \beta_j = \alpha_{j+1} \lambda_{j+1} + \beta_{j+1}$;

and $u : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$u(x) = \begin{cases} \alpha_1 x + \beta_1 & \text{if} \quad \lambda_1 \leq x \leq \lambda_2 \\ \alpha_2 x + \beta_2 & \text{if} \quad \lambda_2 \leq x \leq \lambda_3 \\ \cdots & \cdots \qquad \cdots \\ \alpha_k x + \beta_k & \text{if} \qquad \lambda_k \leq x \end{cases}$$

[1]The main mathematical conventions and notations, used here, are as follows. The set of non-negative (positive) real numbers is $\mathbb{R}_+$ ($\mathbb{R}_{++}$). Vector inequalities are denoted by $>$ and $\geq$. More precisely, $x > y$ means that each coordinate of x is greater than the corresponding coordinate of y, whereas $x \geq y$ allows some of them to be equal. Finally, given a set N and a subset M, we denote the projection of the vector $v \in \mathbb{R}_+^{|N|}$ over M as v_M, i.e., $v_M = (v_i)_{i \in M}$.

for each $x \in \mathbb{R}_+$, then $u \in \mathcal{U}$.[2] Similarly, if $u \in \mathcal{U}$, $c \in \mathbb{R}_+$ and $v : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by $v(x) = u(x + c)$, for each $x \in \mathbb{R}_+$, then $v \in \mathcal{U}$. Note that $u_i(0) \geq 0$ denotes the outcome that agent i can generate with zero wealth. These levels, to be interpreted as agents' starting points, may well differ. In the example of life expectancy as a function of investment in health care, mentioned above, this would mean that even though some agents might not survive without investment in their health care, some others might do so.

Let $\mathcal{I}$ be the family of all finite subsets of $\mathbb{I}$. We define an **economy** e as a triple (N, u, W), where $N = \{i_1, i_2, ..., i_n\} \in \mathcal{I}$ is the set of agents, $u = (u_i)_{i \in N}$ is the profile of their outcome functions (defined as above), and $W \in \mathbb{R}_+$ represents the available wealth. The family of all economies is $\mathcal{E}$. The following subfamily of economies is worth defining for the ensuing discussion.

$$\mathcal{E}^0 \equiv \{e = (N, u, W) \in \mathcal{E} : u_i(0) = 0 \text{ for each } i \in N\}.$$

In words, $\mathcal{E}^0$ is the subfamily of economies in which the starting points of all agents are null.

An *allocation rule*, or simply a *rule*, is a function R that associates with each economy an allocation indicating how to distribute the wealth available in the economy among its members. Formally, $R : \mathcal{E} \to \mathbb{R}^n_+$, where, for each $e = (N, u, W) \in \mathcal{E}$, $R(e) = (R_i(e))_{i \in N} \in \mathbb{R}^n_+$ is such that $\sum_{i \in N} R_i(e) = W$. As we discard all information that is not contained in the description of an economy, we implicitly assume that rules are anonymous. In other words, the identity of agents will not play a role in the allocation process and we shall only focus on the outcome functions and the available wealth of the economy.

Examples of rules are the following. First, the rule that awards each agent the same amount:

***Resource-Egalitarian** rule (RE):* $RE_i(N, u, W) = \frac{W}{n}$.

An alternative to the resource-egalitarian rule is obtained by focusing on the levels of outcome agents achieve, as opposed to the resources they receive, and choosing the vector at which these outcome levels are as equal as possible. In other words, the wealth is allocated initially to the agent(s) with the lowest starting point until her (their) outcome(s) become equal to the starting point(s) of the agent(s) with the second lowest starting point. Then, the rest of the wealth is distributed between these agents in a way to equalize their outcomes, until their outcomes are equal to the starting point(s) of the agent(s) with the third lowest starting point and so on. Formally,

[2] In particular, this implies that the graphs of all admissible outcome functions will cover the positive orthant, as well as the non-negative vertical axis.

Constrained Outcome-Egalitarian *rule (COE):* $COE_i(N,u,W) = u_i^{-1}(\max\{\lambda, u_i(0)\})$, *where* $\lambda \geq 0$ *is chosen so that* $\sum_{i \in N} u_i^{-1}(\max\{\lambda, u_i(0)\}) = W$.

In other words, for each economy (N,u,W), the constrained outcome-egalitarian rule determines a set $N_1 \subseteq N$ such that, for each $i,j \in N_1$ and $k \in N \setminus N_1$,

$$u_i(COE_i(N,u,W)) = u_j(COE_j(N,u,W)) \leq u_k(0) = u_k(COE_k(N,u,W)).$$

These two rules, which naturally translate the focal egalitarian theories we alluded to in the introduction, will be salient in our analysis. Nevertheless, many other rules can be defined. Instances would be dictatorial rules awarding a single agent the whole amount of wealth, proportional-like rules awarding wealth proportionally to some individual characteristic, or compromises between the above two arising from equalizing some combination between resources and outcome levels, as described next.

Formally, let Φ be the family of all functions $\varphi : \{\mathbb{R}_+ \times \mathbb{R}_{++}\} \cup \{(0,0)\} \to \mathbb{R}_+$ satisfying the following monotonicity and continuity assumptions:

Monotonicity:

(i) If $x \in \mathbb{R}_+$ and $x', y \in \mathbb{R}_{++}$ are such that $x < x'$ then $\varphi(x,y) \leq \varphi(x',y)$.

(ii) If $x \in \mathbb{R}_+$ and $y, y' \in \mathbb{R}_{++}$ are such that $y < y'$ then $\varphi(x,y) \leq \varphi(x,y')$.

(iii) If $x \in \mathbb{R}_+$ and $x', y, y' \in \mathbb{R}_{++}$ are such that $(x,y) < (x',y')$, then $\varphi(x,y) < \varphi(x',y')$.

(iv) $\varphi(0,0) = 0 \leq \varphi(x,y)$ for each $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_{++}$.

Continuity:

(i) φ is continuous on $\mathbb{R}^2_{++}$, i.e.,

$$\lim_{(x,y)\to(x_0,y_0)} \varphi(x,y) = \varphi(x_0,y_0), \text{ for each } (x_0,y_0) \in \mathbb{R}^2_{++}.$$

(ii) For each $y_0 \in \mathbb{R}_+$, $\varphi(x,y)$ approaches $\varphi(0,y_0)$, as (x,y) approaches $(0,y_0)$ along any continuous and strictly increasing curve $y = y(x)$ in $\mathbb{R}^2_{++}$, i.e.,

$$\lim_{x\to 0^+} \varphi(x,y(x)) = \varphi(0,y_0),$$

for each $y : \mathbb{R}_{++} \to \mathbb{R}_{++}$ continuous and strictly increasing function such that $\lim_{x\to 0^+} y(x) = y_0$.

Let φ be a function in the class Φ. For each $i \in \mathbb{I}$ define the function $\psi_i : \mathbb{R}_+ \to \mathbb{R}_+$ by $\psi_i(w) = \varphi(w, u_i(w))$ for each $w \in \mathbb{R}_+$. Given the monotonicity and continuity conditions of φ, it follows that, for each $i \in \mathbb{I}$, ψ_i is continuous and strictly increasing. Then, we define the family of *constrained index-egalitarian* rules as follows:

Constrained Index-Egalitarian *rules* $(\{E^{\varphi}\}_{\varphi \in \Phi})$: $E_i^{\varphi}(N,u,W) = \psi_i^{-1}(\max\{\xi, \psi_i(0)\})$, *where* $\xi > 0$ *is chosen so that* $\sum_{i\in N} \psi_i^{-1}(\max\{\xi, \psi_i(0)\}) = W$.

In words, for each $\varphi \in \Phi$, and $e \in \mathcal{E}$, $E^{\varphi}(e)$ is the wealth allocation that lexicographically maximizes the φ-*value* across agents in e. More precisely, if agents are ranked according to the φ-*value* from their starting points, then the wealth is allocated initially to the agent with the lowest one until the resulting φ-*value* becomes equal to the second lowest (starting point) φ-*value*. Then, the rest of the wealth is distributed between these two agents in a way to equalize their φ-*values*, until these are equal to the third lowest (starting point) φ-*value* and so on. Equivalently, for each economy (N,u,W), the constrained index-egalitarian rule determines a set $N_1 \subseteq N$ such that, for each $i,j \in N_1$ and $k \in N \setminus N_1$,

$$\psi_i(E_i^{\varphi}(N,u,W)) = \psi_j(E_j^{\varphi}(N,u,W)) \leq \psi_k(0) = \psi_k(E_k^{\varphi}(N,u,W)).$$

Note that, applied in this manner to an agent's wealth and outcome, φ can be considered as a generalized index of wealth and outcome. So the rules just defined leximin a generalized index of wealth and outcome. It is straightforward to show that each rule E^{φ} actually equalizes the corresponding index φ, when restricted to economies in $\mathcal{E}^0$. Hence, this family is a generalization to this context of the family introduced in Moreno-Ternero and Roemer (2006).[3] It is also straightforward to show that the resource-egalitarian rule and the constrained outcome-egalitarian rule are members of the family of constrained index-egalitarian rules. Formally, if $\varphi(x,y) = x$, for each $(x,y) \in \mathbb{R}^2_+$, then $E^{\varphi} \equiv RE$, whereas if $\varphi(x,y) = y$, for each $(x,y) \in \mathbb{R}^2_+$, then $E^{\varphi} \equiv COE$.

We now present several axioms for allocation rules that we endorse. We begin by introducing our axiom of *priority*, which says that no agent can dominate another agent both in resources and outcome.

Priority *(PR). Let* $e = (N,u,W) \in \mathcal{E}$ *and* $i,j \in N$ *be such that* $R_i(e) < R_j(e)$. *Then* $u_i(R_i(e)) \geq u_j(R_j(e))$.

Note that this axiom (first introduced under this form in Moreno-Ternero and Roemer, 2006) guarantees that agents with poor outcome functions receive at least as much wealth as agents with better outcome functions. In other words, priority implies the *weak equity* axiom, introduced by Sen (1973). On the other hand, priority also says that an agent with a poor

[3]The model in Moreno-Ternero and Roemer (2006) did not allow for differences in starting points and, therefore, the analysis therein was restricted to the domain of economies $\mathcal{E}^0$.

outcome function is never allocated so much that her outcome level exceeds that of an agent with a better outcome function.[4]

Our next axiom, *solidarity*, says that the arrival of new agents, whether or not this is accompanied by a change in the available wealth, should affect all the incumbent agents in the same direction. Formally,

Solidarity *(SL). Let* $e = (N, u, W) \in \mathcal{E}$ *and* $e' = (N', u', W') \in \mathcal{E}$*, such that* $N \subseteq N'$*. Let* $N_1 = \{i \in N : R_i(e) > 0\}$*. Then, one of the following three statements holds:*

$$R_i(e') = R_i(e), \text{ for each } i \in N,$$

$$R_i(e') \geq R_i(e), \text{ for each } i \in N, \text{ and } R_i(e') > R_i(e), \text{ for each } i \in N_1,$$

$$R_i(e') \leq R_i(e), \text{ for each } i \in N, \text{ and } R_i(e') < R_i(e), \text{ for each } i \in N_1.$$

Our solidarity axiom is modeling the fact that agents cannot benefit from a change (either in the available wealth or in the number of agents) if someone else suffers from it. Related formulations of the solidarity notion abound in the literature (see Thomson (forthcoming) for a survey, and Maniquet and Sprumont (2010) for a recent instance).[5] This axiom is equivalent to the combination of two axioms that appear frequently in the literature: *resource monotonicity* and *consistency*. Resource monotonicity (e.g., Roemer, 1986) says that when a bad or good shock comes to an economy, all its members should share in the calamity or windfall.

Resource monotonicity *(RM). Let* $e = (N, u, W)$ *and* $e' = (N, u, W') \in \mathcal{E}$ *be such that* $W' < W$*. Let* $N_1 = \{i \in N : R_i(e) > 0\}$*. Then,* $R_i(e') \leq R_i(e)$ *for each* $i \in N$*, and* $R_i(e') < R_i(e)$ *for each* $i \in N_1$*.*

Consistency has played a fundamental role in axiomatic analysis (see, e.g., Thomson (2007) and the literature cited therein) even though it is mostly an operational (rather than ethical) axiom. It says that if a sub-group of agents secedes with the resource allocated to it under R then, in the smaller economy, R allocates the resource in the same way. In that sense, consistency can be interpreted as a notion of stability.

[4] A counterpart to this axiom in the theory of fairness is the so-called "no-domination" axiom (e.g., Thomson, 2011), which says that no agent receives more of all goods than some other agent.

[5] Solidarity properties with respect to population changes were indeed introduced by Thomson (1983a,b). As mentioned above, our axiom is stronger as it refers to simultaneous changes in two parameters (namely, resources and population). Chun (1999) formulates a counterpart notion in the context of bankruptcy problems.

Consistency *(CY). Let* $e = (N, u, W)$ *and* $e' = (N', u', W') \in \mathcal{E}$ *be such that* $N' \subset N$ *and* $W' = \sum_{i \in N'} R_i(e)$*. Then* $R_i(e) = R_i(e')$*, for each* $i \in N'$*.*

Our final property pertains to the behavior of a rule with respect to tentative allocations based on an incorrect estimation of the available wealth. To motivate this property, imagine the following scenario: after having committed to divide the available wealth, one finds that the actual amount to divide is larger than was initially assumed. Then, two options are open: either the tentative division is cancelled altogether and the allocation for the actual economy is obtained directly, or we add to the initial commitment the result of applying the rule to the subsequent economy with the remaining amount and the adjusted individual outcome functions that would emerge after the initial allocation.[6] The requirement of *composition* is that both ways of proceeding should result in the same allocations. Formally,

Composition *(CP). Let* $e = (N, u, W) \in \mathcal{E}$*. Let* $W^1, W^2 \in \mathbb{R}_{++}$ *be such that* $W = W^1 + W^2$ *and* $e^1 = (N, u, W^1) \in \mathcal{E}$*. For each* $i \in N$*, let* $\hat{i} \in \mathbb{I}$ *be such that* $\widehat{u}_{\hat{i}}(x) = u_i(x + R_i(e^1))$ *for each* $x \in \mathbb{R}_+$*, and let* $e^2 = \left(\widehat{N}, (\widehat{u}_{\hat{i}})_{\hat{i} \in \widehat{N}}, W^2\right) \in \mathcal{E}$*. Then,* $R(e) = R(e^1) + R(e^2)$*.*

The property of composition has a relative in the theory of axiomatic bargaining: the so-called "step-by-step negotiations" axiom introduced by Kalai (1977), which considers two nested bargaining sets and uses the bargaining solution for the smaller bargaining set as the disagreement point for the larger bargaining set. It is the basis for the characterization of the egalitarian solution in such context. The same principle has also been frequently used in other related contexts (e.g., Young, 1988; Moulin, 2000; Moulin and Stong, 2002).

3 The results

As anticipated in the introduction, the main result of this paper shows that the resource-egalitarian rule and the constrained outcome-egalitarian rule are characterized by the axioms described above. In order to prove such result, we present first another result, which generalizes Theorem 1 in Moreno-Ternero and Roemer (2006).

Theorem 1 *A rule defined on* $\mathcal{E}$ *satisfies priority and solidarity if and only if it is a constrained index-egalitarian rule.*

The proof of Theorem 1 appears in the Appendix. Note that this theorem is indeed a generalization of Theorem 1 in Moreno-Ternero and Roemer (2006), which says that, for the domain

[6] This is the reason why we require $\mathcal{U}$ to be closed under horizontal translations.

of economies $\mathcal{E}^0$, the family of index-egalitarian rules is characterized by the corresponding axiom of restricted domain, the axiom of priority and the solidarity axiom.

We are then ready to state our main result.

Theorem 2 *A rule defined on $\mathcal{E}$ satisfies priority, solidarity and composition if and only if it is either the resource-egalitarian rule or the constrained outcome-egalitarian rule.*

Proof. By Theorem 1, we know that RE and COE satisfy PR and SL. It is straightforward to show that RE satisfies CP. We then show that COE satisfies CP.

Let $e = (N, u, W) \in \mathcal{E}$. For ease of exposition, assume that $N = \{1, 2, ..., n\}$ and that agents are ranked (in an increasing order) according to their initial starting points, i.e., $u_i(0) \leq u_{i+1}(0)$ for each $i = 1, ..., n-1$. Let $W^1, W^2 \in \mathbb{R}_{++}$ be such that $W = W^1 + W^2$ and let $e^1 = (N, u, W^1) \in \mathcal{E}$. For each $i \in N$, let $\hat{i} \in \mathbb{I}$ be such that $\widehat{u}_{\hat{i}}(x) = u_i(x + COE_i(e^1))$ for each $x \in \mathbb{R}_+$, and let $e^2 = (\widehat{N}, \widehat{u}, W^2) \in \mathcal{E}$, where $\widehat{u} = (\widehat{u}_{\hat{i}})_{i \in N}$. Let $\sigma_i = u_i^{-1}$ and $\widehat{\sigma}_{\hat{i}} = \widehat{u}_{\hat{i}}^{-1}$ for each $i \in N$. Then,

$$COE_i(e) = \begin{cases} \sigma_i(\lambda) & \text{for each } i = 1, \ldots, k \\ 0 & \text{for each } i = k+1, \ldots, n \end{cases}$$

where λ and k are such that

$$\sum_{i=1}^{k} \sigma_i(\lambda) = W, \text{ and } u_{k+1}(0) > \lambda \geq u_k(0).$$

Similarly,

$$COE_i(e^1) = \begin{cases} \sigma_i(\lambda_1) & \text{for each } i = 1, \ldots, k_1 \\ 0 & \text{for each } i = k_1+1, \ldots, n \end{cases}$$

where λ_1 and k_1 are such that

$$\sum_{i=1}^{k_1} \sigma_i(\lambda_1) = W^1, \text{ and } u_{k_1+1}(0) > \lambda_1 \geq u_{k_1}(0).$$

Thus, note that $k \geq k_1$ and $\lambda \geq \lambda_1$. Finally,

$$COE_i(e^2) = \begin{cases} \widehat{\sigma}_{\hat{i}}(\lambda_2) & \text{for each } i = 1, \ldots, k_2 \\ 0 & \text{for each } i = k_2+1, \ldots, n \end{cases}$$

where λ_2 and k_2 are such that

$$\sum_{i=1}^{k_2} \widehat{\sigma}_{\hat{k}_2}(\lambda_2) = W^2, \text{ and } \widehat{u}_{\widehat{k_2+1}}(0) > \lambda_2 \geq \widehat{u}_{\hat{k}_2}(0).$$

Let $y = COE(e) - COE(e^1)$ and $z = COE(e^2)$. We have to show that $y = z$. To do so, note first that

$$\widehat{u}_{\hat{i}}(y_i) = \begin{cases} \lambda & \text{for each } i = 1, \ldots, k \\ u_i(0) & \text{for each } i = k+1, \ldots, n \end{cases}$$

and

$$\widehat{u}_{\hat{i}}(z_i) = \begin{cases} \lambda_2 & \text{for each } i = 1, \ldots, k_2 \\ \widehat{u}_{\hat{i}}(0) & \text{for each } i = k_2+1, \ldots, n \end{cases}$$

Note also that, as y is a feasible allocation for the economy e^2, it follows, by definition of COE, that $(\widehat{u}_{\hat{i}}(z_i))_{i \in N}$ lexicographically dominates $(\widehat{u}_{\hat{i}}(y_i))_{i \in N}$. This implies that $\lambda \leq \lambda_2$. Thus, $k_1 \leq k \leq k_2$. Then,

$$y_i = \begin{cases} \sigma_i(\lambda) - \sigma_i(\lambda_1) & \text{for each } i = 1, \ldots, k_1 \\ \sigma_i(\lambda) & \text{for each } i = k_1+1, \ldots, k \\ 0 & \text{for each } i = k+1, \ldots, n \end{cases}$$

and

$$z_i = \begin{cases} \widehat{\sigma}_{\hat{i}}(\lambda_2) & \text{for each } i = 1, \ldots, k_2 \\ 0 & \text{for each } i = k_2+1, \ldots, n \end{cases}$$

Let $i = 1, \ldots, k_1$. Then, $\widehat{u}_{\hat{i}}(x) = u_i(x + \sigma_i(\lambda_1))$ for each $x \in \mathbb{R}_+$. Thus, $\widehat{\sigma}_{\hat{i}}(x) = \sigma_i(x) - \sigma_i(\lambda_1)$ for each $x \in \mathbb{R}_+$. In particular, $\widehat{\sigma}_{\hat{i}}(\lambda_2) = \sigma_i(\lambda_2) - \sigma_i(\lambda_1)$. Similarly, $\widehat{\sigma}_{\hat{i}}(\lambda_2) = \sigma_i(\lambda_2)$ for each $i = k_1+1, \ldots, k_2$. Thus, $z_i \geq y_i$ for each $i \in N$. Now, if $\lambda < \lambda_2$, we would have $W^2 = \sum_{i \in N} z_i > \sum_{i \in N} y_i = W^2$, a contradiction. Thus, it follows that $\lambda = \lambda_2$ and, therefore, that $k = k_2$, which implies that $y = z$, as desired.

We conclude by showing that no other rule within the family $\{E^{\varphi}\}_{\varphi \in \Phi}$ satisfies CP.

Let $\widehat{\Phi}$ denote the residual of Φ after removing the functions giving rise to RE and COE. We partition the family $\widehat{\Phi}$ according to the following concept. We say that $\varphi \in \widehat{\Phi}$ is *quasilinear in* x if there exists $\lambda > 0$ and $f : \mathbb{R}_+ \to \mathbb{R}_+$, continuous and increasing, with $f(0) = 0$, and $f(x) > 0$ for some $x > 0$, such that $\varphi(x, y) = \lambda x + f(y)$, for each $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_{++}$.[7]

Case 1. The quasiliniear case.

Let $\varphi \in \widehat{\Phi}$ be a quasilinear function in x. Then, there exists $y_2 \in \mathbb{R}_{++}$ such that $\varphi(0, y_2) > 0$. Furthermore, there exist $\delta \in \mathbb{R}_{++}$ and $(x_1, y_1) \in \mathbb{R}^2_{++}$, with $x_1 > \delta$ and $y_2 > y_1$, such that $\varphi(x_1 - \delta, y_1) = \varphi(0, y_2) > 0$.

[7]In other words, the level curves of φ are parallel displacements of each other along the x axis.

Let $u_1, u_2 \in \mathcal{U}$ be such that

$$u_1(0) = 0, \quad u_2(0) = y_2, \quad u_1(x_1) = y_1.$$

Let $\lambda = \psi_1(x_1) = \varphi(x_1, y_1) > 0$ and $W = \psi_1^{-1}(\lambda) + \psi_2^{-1}(\lambda)$ and consider the economy $e = (\{1,2\}, (u_1, u_2), W)$. Then, it is straightforward to show that

$$E^{\varphi}(e) = \left(\psi_1^{-1}(\lambda), \psi_2^{-1}(\lambda)\right).$$

Let $\varepsilon \in \mathbb{R}_{++}$ be such that $\varepsilon < W$ and $\varphi(\varepsilon, u_1(\varepsilon)) < \varphi(0, y_2)$ and consider the economy $e^1 = (\{1,2\}, (u_1, u_2), \varepsilon)$. Then, it is straightforward to show that

$$E^{\varphi}(e^1) = (\varepsilon, 0).$$

Finally, let $W_2 = W - \varepsilon > 0$. For $i = 1, 2$, let $\widehat{u}_{\hat{i}} : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $\widehat{u}_{\hat{i}}(x) = u_i(x + E_i^{\varphi}(e^1))$ and consider the economy $e^2 = (\{\widehat{1}, \widehat{2}\}, (\widehat{u}_{\hat{1}}, \widehat{u}_{\hat{2}}), W^2)$. For $i = 1, 2$, let $\hat{\psi}_i : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $\hat{\psi}_i(w) = \varphi(w, \widehat{u}_{\hat{i}}(w))$ for each $w \in \mathbb{R}_+$.

Assume, by contradiction, that E^{φ} satisfies CP. Then, if $E^{\varphi}(e^2) = (W_2, 0)$, it follows that $\psi_2^{-1}(\lambda) = 0$, which would imply $\varphi(x_1, y_1) = \lambda = \varphi(0, y_2) = \varphi(x_1 - \delta, y_1)$, a contradiction with the fact that φ is quasilinear. If, on the other hand, $E^{\varphi}(e^2) = \left(\hat{\psi}_1^{-1}(\lambda'), \hat{\psi}_2^{-1}(\lambda')\right)$, then $\psi_2^{-1}(\lambda) = \hat{\psi}_2^{-1}(\lambda')$ and $\psi_1^{-1}(\lambda) = \varepsilon + \hat{\psi}_1^{-1}(\lambda')$. From the former equality, it follows that $\lambda = \lambda'$, as $\hat{\psi}_2 \equiv \psi_2$ is a strictly increasing function. Thus, from the latter equality, it follows that $x_1 = \varepsilon + \hat{\psi}_1^{-1}(\lambda)$, or, equivalently, $\varphi(x_1, y_1) = \lambda = \hat{\psi}_1(x_1 - \varepsilon) = \varphi(x_1 - \varepsilon, u_1(x_1)) = \varphi(x_1 - \varepsilon, y_1)$, again, a contradiction with the fact that φ is quasilinear.

Case 2. The non-quasiliniear case.

Let $\varphi \in \widehat{\Phi}$ be a non-quasilinear function in x. Then, there exist $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2_{++}$, and $0 < \alpha < \min\{x_1, x_2\}$, such that $\varphi(x_1, y_1) = \varphi(x_2, y_2)$ and $\varphi(x_1 - \alpha, y_1) \neq \varphi(x_2 - \alpha, y_2)$.

Let $u_1, u_2 \in \mathcal{U}$ be such that

$$u_1(x_1) = y_1, \quad u_2(x_2) = y_2, \quad u_1(\alpha) = u_2(\alpha).$$

Now, consider the economies $e^1 = (\{1,2\}, (u_1, u_2), 2\alpha)$ and $e = (\{1,2\}, (u_1, u_2), x_1 + x_2)$. It is straightforward to show that

$$E^{\varphi}(e^1) = (\alpha, \alpha),$$

and

$$E^{\varphi}(e) = (x_1, x_2).$$

For $i = 1,2$, let $\widehat{u}_{\hat{i}} : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $\widehat{u}_{\hat{i}}(x) = u_i(x+\alpha)$ and consider the economy $e^2 = (\{\widehat{1}, \widehat{2}\}, (\widehat{u}_{\hat{1}}, \widehat{u}_{\hat{2}}), W^2)$, where $W^2 = x_1 + x_2 - 2\alpha$. For $i = 1,2$, let $\hat{\psi}_i : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $\hat{\psi}_i(w) = \varphi(w, \widehat{u}_{\hat{i}}(w))$ for each $w \in \mathbb{R}_+$. Then,

$$E^{\varphi}(e^2) = \left(\hat{\psi}_1^{-1}(\lambda'), \hat{\psi}_2^{-1}(\lambda')\right),$$

where λ' is such that $\psi_1^{-1}(\lambda') + \psi_2^{-1}(\lambda') = W^2$. Equivalently,

$$E^{\varphi}(e^2) = (x, x_1 + x_2 - 2\alpha - x),$$

where

$$\varphi(x, u_1(x+\alpha)) = \varphi(x_1 + x_2 - 2\alpha - x, u_2(x_1 + x_2 - \alpha - x)). \tag{1}$$

Now, $E^{\varphi}(e) = E^{\varphi}(e^1) + E^{\varphi}(e^2)$ if and only if $\alpha + x = x_1$. But if so, (1) becomes

$$\varphi(x_1 - \alpha, y_1) = \varphi(x_2 - \alpha, y_2),$$

which represents a contradiction. ■

4 Related literature

Our paper can be considered as part of the rapidly expanding literature on fair allocation. Traditionally, economists have been criticized for paying too little attention to distributional questions. There now exists, however, a well-developed literature devoted to the formulation and the analysis of equity concepts that traces back to Foley (1967) and his notion of envy-free allocation. In the last few years, a variety of new solutions has been proposed and applied to a wide range of models, and a number of properties of solutions have been formulated and studied for these models (see, for instance, Fleurbaey and Maniquet, 2011; Thomson, 2011; and the literature cited therein). To a large extent, this literature has been axiomatic, taking as the departure properties of allocation rules and investigating the existence of rules satisfying various combinations of these properties. This is precisely what we do in this paper for a model of resource allocation in which agents' capabilities and starting points may differ.

The use of the axiomatic method is not a discovery of the theory of fair allocation but of the theory of bargaining initiated by Nash (1950). Bargaining theory is the axiomatic study of utility-allocating mechanisms acting on a domain of utility possibility sets with threat points. The applications of bargaining theory to problems of distributive justice are usually not supportable, because too much justice-relevant information is lost in the specification of the domain. This has motivated the extension of the theory to economic environments, in which an aggregate starting point vector has to be allocated between several agents with given utility functions. Our model could then be considered as a specific case in which the good to be allocated is unidimensional. This seemingly innocuous aspect makes, however, our analysis independent from the (extended) bargaining theory with economic environments, in which a key assumption is that the domain consists of all environments with any positive number of dimensions (e.g., Roemer, 1988).

Axioms similar to the ones used in this paper have also been used in different, but related, models in the literature. Instances are the division problem with single-peaked preferences (e.g., Sprumont, 1991), the taxation model (e.g., Young, 1988), the rationing model (e.g., Moulin, 2000), and the scheduling model (e.g., Moulin and Stong, 2002). In those models, peaks, incomes, claims, or demands of the agents play a crucial role in resource allocation. In our case, the starting points of agents play a sort of counterpart role. The constrained outcome-egalitarian rule studied in this paper resembles, to some extent, the uniform rule, the constrained equal awards rule, and "standard of gains" method in those models. Moulin and Stong (2002), for instance, characterize the latter by consistency, composition, and demand monotonicity (a counterpart of this axiom in the current setting might be: as starting points increase, allotments do not decrease). Hence, in addition to the common axioms, Moulin and Stong (2002) use the equity axiom of "demand monotonicity", whereas here "priority" is used instead. Somewhat relatedly, Young (1988) characterizes in the taxation model the so-called equal-sacrifice solutions by strict resource-monotonicity, composition, consistency, and strict order-preservation.

5 Further insights

We have analyzed a simple distribution problem in which a given amount of wealth has to be distributed among individuals possessing a capability to transform wealth into some given valued (interpersonally comparable) outcome and, possibly, different (outcome) starting points. For this simple environment, we have characterized the two focal egalitarian allocation rules that exist, i.e., the one that allocates the resource equally, and the one that makes outcome levels as equal as possible. We have shown that these two focal rules are the only *ethical* and *operational* procedures for allocating wealth, provided we assume that *ethical* means *prioritarian* and *solidaristic*, and *operational* means obeying the axiom of *composition*. In doing so, we have provided, as announced in the introduction, a common ground for the two focal answers to the question "equality of what?". In particular, our characterization result also shows that the combination of the notions of priority, solidarity, and composition is equivalent to a kind of egalitarianism, where the equality in question is either resources or outcome levels.

Our analysis provides a common justification of both resource and welfare egalitarianism. Nevertheless, it is worth mentioning that ours is not a *welfarist* approach, the approach usually linked to welfare egalitarianism and which maintains that the justness of a state should be a function only of the welfares, or outcomes, of the agents in that state. We take instead a *resourcist* approach by studying allocation mechanisms defined on a space of economic environments, where the distribution of resources can be explicitly defined. In other words, we endorse the view that information concerning the distribution of goods or resources is in general necessary to evaluate the justness of a state of the world. As we have seen, this does not preclude our obtaining welfare (in our case, outcome) egalitarianism as a result of combining some (non-welfarist) axioms.[8]

ln moral and political philosophy, the debate between resource egalitarians and welfare egalitarians is between those who wish to hold people responsible for the choices they make and preferences they have, after some initial equality has been guaranteed, and those who wish to hold individuals responsible for nothing about themselves. A natural extension of the model in this paper would account for individual effort decisions. As a matter of fact, the ethical axioms we use in this work would only be justified for equally-deserving individuals. The literature on compensation and responsibility (e.g., Fleurbaey, 2008) provides us with an appropriate framework for such a natural extension. In its simplest case, this literature deals with the allocation of a given amount of an external one-dimensional resource (which is not produced) among a group of individuals whose outcome achievements depend on this resource, but also on their *social background* (a characteristic which elicits compensation) and *personal effort* (a characteristic which does not elicit compensation). In the parlance of our paper, this could be interpreted as saying that individual outcome functions would be bivariate functions depending on two variables reflecting the personal effort of the individual and the amount of the resource she is allocated. The mappings themselves would incorporate the influence of social background on individual outcome achievements. Characterizations of allocation rules for this model exist in the literature. Therefore it would be interesting to explore whether the

[8]An early characterization of welfare egalitarianism, also by means of resourcist axioms on a domain of economic environments, is provided in Roemer (1986).

translation of our axioms to this context would give rise to new characterizations.

Eventually, a theory of distributive justice must, we believe, postulate a domain of economies in which effort choices by individuals (relating to education and production), as well as risk preferences and level-comparable welfare, in a multi-stage model, are described. The present analysis is a far cry from that goal. Indeed, one difficulty in the work of philosophers is that they implicitly assume all these attributes of real-world societies in their theorizing. In any case, it is clear that it would be immensely difficult to deduce formally a theory of just resource allocation on such a domain, without postulating unacceptably strong axioms, and so it is not surprising that the work of political philosophers is tentative and sketchy, by their own admission.

6 Appendix

6.1 The proof of Theorem 1

We start showing that each rule in $\{E^\varphi\}_{\varphi\in\Phi}$ satisfies PR and SL.

Let $\varphi \in \Phi$ and $e = (N, u, W) \in \mathcal{E}$ be given. Let $i, j \in N$ be such that $E_i^\varphi(e) < E_j^\varphi(e)$. Suppose, by contradiction, that $u_i(E_i^\varphi(e)) < u_j(E_j^\varphi(e))$. Then $\psi_i(E_i^\varphi(e)) = \varphi(E_i^\varphi(e), u_i(E_i^\varphi(e))) < \varphi(E_j^\varphi(e), u_j(E_j^\varphi(e))) = \psi_j(E_j^\varphi(e))$, which contradicts the definition of E^φ, as $E_j^\varphi(e) > 0$. It then follows that E^φ satisfies PR.

As for SL, let $\varphi \in \Phi$ be given and let $e = (N, u, W) \in \mathcal{E}$ *and* $e' = (N', u', W') \in \mathcal{E}$, be such that $N \subseteq N'$. Let $N_1 = \{i \in N : R_i(e) > 0\}$.

If $E_i^\varphi(e) = E_i^\varphi(e')$ for each $i \in N$, there is nothing to prove.

Suppose there exists $i \in N$ such that $E_i^\varphi(e') > E_i^\varphi(e)$. Then, $\psi_i(E_i^\varphi(e')) > \psi_i(E_i^\varphi(e))$. As $E_i^\varphi(e') > 0$, it follows that $\psi_j(E_j^\varphi(e)) \leq \psi_i(E_i^\varphi(e)) < \psi_i(E_i^\varphi(e')) \leq \psi_j(E_j^\varphi(e'))$, for each $j \in N_1$. Thus, $E_j^\varphi(e') > E_j^\varphi(e)$, for each $j \in N_1$ and $E_j^\varphi(e') \geq E_j^\varphi(e)$, for each $j \in N$, as desired.

Suppose there exists $i \in N$ such that $E_i^\varphi(e') < E_i^\varphi(e)$. Then, $\psi_i(E_i^\varphi(e')) < \psi_i(E_i^\varphi(e))$. Note that, in this case, $i \in N_1$. Then, $\psi_j(E_j^\varphi(e)) = \psi_i(E_i^\varphi(e)) > \psi_i(E_i^\varphi(e'))$, for each $j \in N_1$. Assume that $E_j^\varphi(e') > 0$ (otherwise, there is nothing to prove). Then, $\psi_j(E_j^\varphi(e)) > \psi_i(E_i^\varphi(e')) \geq \psi_j(E_j^\varphi(e'))$ and, therefore, $E_j^\varphi(e') < E_j^\varphi(e)$, as desired. To conclude, suppose, by contradiction, that there exists $j \in N \setminus N_1$ such that $E_j^\varphi(e') > E_j^\varphi(e) = 0$. Then, $\psi_j(E_j^\varphi(e)) \geq \psi_i(E_i^\varphi(e)) > \psi_i(E_i^\varphi(e')) = \psi_j(E_j^\varphi(e'))$, which contradicts the monotonicity of ψ_j.

We now show that if a rule R satisfies PR and SL then it is a member of the family $\{E^\varphi\}_{\varphi\in\Phi}$.

Let R be a rule satisfying PR and SL. By Theorem 1 in Moreno-Ternero and Roemer (2006) there is a continuous and non-decreasing index $\varphi : \mathbb{R}^2_{++} \cup \{(0,0)\} \rightarrow \mathbb{R}_+$, such that $\inf\{\varphi(x,y)\} = \varphi(0,0) = 0$ and for each $(x,y) > (z,t)$, $\varphi(x,y) > \varphi(z,t)$, for which[9]

$$R(e) = E^{\varphi}(e) \text{ for each } e \in \mathcal{E}^0.$$

Without loss of generality, let us denote by 1 the agent in $\mathbb{I}$ whose outcome function is defined by $u_1(x) = x$, for each $x \in \mathbb{R}_+$. The following claim, which holds thanks to PR and SL, will be used throughout the proof.

Claim. Let $e = (\{1,l\}, (u_1, u_l), W) \in \mathcal{E}^0$ and $R(e) = (W - \hat{x}, \hat{x})$. Then, $R(e^j) = R(e)$ for each $e^j = (\{1,j\}, (u_1, u_j), W) \in \mathcal{E}$, in which $j \in \mathbb{I}$ is such that $u_j(\hat{x}) = u_l(\hat{x})$.

For each $y \in \mathbb{R}_+$, let

$$\Omega^y = \{W \in \mathbb{R}_{++} : R_j(e) > 0, e = (\{1,j\}, (u_1, u_j), W), j \in \mathbb{I}, u_j(0) = y\}.$$

We extend the definition of φ to include the set $\{0\} \times \mathbb{R}_{++}$ on its domain, as follows:

$$\varphi(0,y) = \inf \Omega^y \text{ for each } y \in \mathbb{R}_{++}.$$

We show first that $\varphi : \{\mathbb{R}_+ \times \mathbb{R}_{++}\} \cup \{(0,0)\} \rightarrow \mathbb{R}_+$, so extended, belongs to Φ. Then, to conclude the proof of the theorem, we show that $R(e) = E^{\varphi}(e)$ for each $e \in \mathcal{E} \setminus \mathcal{E}^0$.

- φ exhibits the following monotonicity properties:

  (i) If $x \in \mathbb{R}_+$ and $x', y \in \mathbb{R}_{++}$ are such that $x < x'$ then $\varphi(x,y) \leq \varphi(x',y)$.

  (ii) If $x \in \mathbb{R}_+$ and $y, y' \in \mathbb{R}_{++}$ are such that $y < y'$ then $\varphi(x,y) \leq \varphi(x,y')$.

  (iii) If $x \in \mathbb{R}_+$ and $x', y, y' \in \mathbb{R}_{++}$ are such that $(x,y) < (x',y')$, then $\varphi(x,y) < \varphi(x',y')$.

  (iv) $\varphi(0,0) = 0 \leq \varphi(x,y)$ for all $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_{++}$.

Given the monotonicity properties of φ on $\mathbb{R}^2_{++}$ described above, it suffices to prove the first two items when $x = 0$ in each of them.

Let $x', y \in \mathbb{R}_{++}$ and $\alpha = \varphi(x', y)$. Assume, without loss of generality, that $x' < y$. Let $k \in \mathbb{I}$ be an agent whose outcome function satisfies that $u_k(0) = 0$ and $u_k(x') = y$. Then, consider the economy $e_k = (\{1,k\}, (u_1, u_k), \alpha + x')$. As $e_k \in \mathcal{E}^0$, it follows that $R(e_k) = E^{\varphi}(e_k) = (\alpha, x')$. By

[9] Without loss of generality, we can assume that $\varphi(x,x) = x$, for each $x \in \mathbb{R}_{++}$.

PR, $x' \leq \alpha \leq y$. Let $j \in \mathbb{I}$ be an agent whose outcome function satisfies that $u_j(0) = y$. As the iso-level sets of the index φ on $\mathbb{R}^2_{++}$ slope downward to the right, there exists $(\bar{x}, \bar{y}) \in \mathbb{R}^2_{++}$, with $x' \leq \bar{x}$ and $y \geq \bar{y}$, such that $\varphi(\bar{x}, \bar{y}) = \varphi(x', y) = \alpha = u_j(\bar{x})$. Assume, without loss of generality, that $(\bar{x}, \bar{y}) \neq (0, y)$. Let $l \in \mathbb{I}$ be an agent whose outcome function satisfies that $u_l(0) = 0$ and $u_l(\bar{x}) = u_j(\bar{x})$ and consider the economy $e_l = (\{1, l\}, (u_1, u_l), \alpha + \bar{x})$. As $e_l \in \mathcal{E}^0$, it follows that $R(e_l) = E^{\varphi}(e_l) = (\alpha, \bar{x})$. Finally, consider the economy $e_j = (\{1, j\}, (u_1, u_j), \alpha + \bar{x})$. By the above claim, $R(e_j) = R(e_l) = (\alpha, \bar{x})$. By RM, $R_j(e^*_j) > 0$, where $e^*_j = (\{1, j\}, (u_1, u_j), \alpha)$. Thus, $\alpha = \varphi(x', y) \in \Omega^y$ and, therefore, $\varphi(0, y) = \inf \Omega^y \leq \varphi(x', y)$, as desired.

As for statement (ii), let $y, y' \in \mathbb{R}_{++}$ be such that $y < y'$. We show next that $\Omega^{y'} \subset \Omega^y$, from where it would follow that $\varphi(0, y) = \inf \Omega^y \leq \inf \Omega^{y'} = \varphi(0, y')$, as desired. Let $W \in \Omega^{y'}$. Then, there exists $k \in \mathbb{I}$ such that $u_k(0) = y'$ and for which $R_k(e_k) > 0$, where $e_k = (\{1, k\}, (u_1, u_k), W)$. Now, let $j \in \mathbb{I}$ be such that $u_j(0) = y$ and consider the economy $e_j = (\{1, j\}, (u_1, u_j), W)$. It suffices to show that $R_j(e_j) > 0$. Suppose otherwise, and let $e = (\{1, j, k\}, (u_1, u_j, u_k), W)$. Then, by SL applied to e_j and e, it follows that $R_j(e) = 0$. Then, by PR applied to e, it follows that $R_k(e) = 0$ and, therefore, $R(e) = (W, 0, 0)$. Finally, by SL applied to e_k and e, it follows that $R_k(e) = 0$, a contradiction.

- φ exhibits the following continuity properties.

  (i) $\lim_{(x,y)\to(x_0,y_0)} \varphi(x, y) = \varphi(x_0, y_0)$, for all $(x_0, y_0) \in \mathbb{R}^2_{++}$.

  (ii) $\lim_{x\to 0^+} \varphi(x, y(x)) = \varphi(0, y_0)$, for each $y : \mathbb{R}_{++} \to \mathbb{R}_{++}$ continuous and increasing function such that $\lim_{x\to 0^+} y(x) = y_0$.

Only statement (ii) remains to be proved. Let $y_0 \in \mathbb{R}_{++}$ and $\{x_n\} \to 0^+$, with $x_n \in \mathbb{R}_{++}$. Let $y : \mathbb{R}_{++} \to \mathbb{R}_{++}$ be a continuous and increasing function such that $\lim_{x\to 0^+} y(x) = y_0$. For each n, we denote $y_n = y(x_n)$. Our aim is to show that $\{\alpha_n\} = \{\varphi(x_n, y_n)\} \to \alpha = \varphi(0, y_0)$. If such is not the case, then there exists a subsequence $\{\alpha_{k_n}\}$ that converges to $\overline{\alpha} \neq \alpha$. We assume, without loss of generality, that $\overline{\alpha} < \alpha$. Let B be a ball around $(0, y_0)$ and $\widehat{B} = \{(x, y) \in B : x > 0, y > y_0\}$. As $(x_n, y_n) \to (0, y_0)$, it follows that, for large k_n, $(x_{k_n}, y_{k_n}) \in \widehat{B} \cup \{(0, y_0)\}$. Thus, by the monotonicity properties of φ described above, $\varphi(x_{k_n}, y_{k_n}) > \alpha$. On the other hand, as $\alpha_{k_n} \to \overline{\alpha} < \alpha$, it follows that, for large k_n, $\varphi(x_{k_n}, y_{k_n}) = \alpha_{k_n} < \alpha$, which represents a contradiction.

The above shows that $\varphi \in \Phi$. We now conclude the proof of the theorem by showing that $R(e) = E^{\varphi}(e)$ for each $e \in \mathcal{E} \setminus \mathcal{E}^0$. By the above claim, and the fact that R coincides with E^{φ} on

$\mathcal{E}^0$, it follows that coincidence also holds for the domain of two-agent economies (in $\mathcal{E}$) involving agent 1. To conclude, let us consider a general economy $e = (N, u, W) \in \mathcal{E}$. By RM and PR, there is $W' > W$ such that $R_1(e') = W' - W$, where $e' = (\{1\} \cup N, (u_1, u), W')$. Let $x = R_N(e')$ and, for each $j \in N$, $e_j = (\{1, j\}, (u_1, u_j), W' - W + x_j)$. By CY, $R(e_j) = R_{\{1,j\}}(e') = (W' - W, x_j)$. Now, as e_j is a two-agent economy involving agent 1, it also follows that $R(e_j) = E^\varphi(e_j)$. Thus, for each $j \in N$, $R(e_j) = E^\varphi(e_j)$, and, therefore, by SL, $E^\varphi(e') = R(e')$. Finally, by CY, $E^\varphi(e) = x = R(e)$, as desired. ■

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To save space, we have included in this appendix, which is not for publication, formal statements and proofs of some aspects that have been dismissed from the body of the paper. In particular, we provide full-fledged versions of the proofs of Theorems 1 and 2, including proofs of some steps that have been dismissed from the body of the paper.

7 Proof of Theorem 1

We start showing that each rule in $\{E^\varphi\}_{\varphi\in\Phi}$ satisfies PR and SL.[10]

Let $\varphi \in \Phi$ and $e = (N, u, W) \in \mathcal{E}$ be given. Let $i, j \in N$ be such that $E_i^\varphi(e) < E_j^\varphi(e)$. Suppose, by contradiction, that $u_i(E_i^\varphi(e)) < u_j(E_j^\varphi(e))$. Then $\psi_i(E_i^\varphi(e)) = \varphi(E_i^\varphi(e), u_i(E_i^\varphi(e))) < \varphi(E_j^\varphi(e), u_j(E_j^\varphi(e))) = \psi_j(E_j^\varphi(e))$, which contradicts the definition of E^φ, as $E_j^\varphi(e) > 0$. It then follows that E^φ satisfies PR.

As for SL, let $\varphi \in \Phi$ be given and let $e = (N, u, W) \in \mathcal{E}$ *and* $e' = (N', u', W') \in \mathcal{E}$, be such that $N \subseteq N'$. Let $N_1 = \{i \in N : R_i(e) > 0\}$.

If $E_i^\varphi(e) = E_i^\varphi(e')$ for each $i \in N$, there is nothing to prove.

Suppose there exists $i \in N$ such that $E_i^\varphi(e') > E_i^\varphi(e)$. Then, $\psi_i(E_i^\varphi(e')) > \psi_i(E_i^\varphi(e))$. As $E_i^\varphi(e') > 0$, it follows that $\psi_j(E_j^\varphi(e)) \leq \psi_i(E_i^\varphi(e)) < \psi_i(E_i^\varphi(e')) \leq \psi_j(E_j^\varphi(e'))$, for each $j \in N_1$. Thus, $E_j^\varphi(e') > E_j^\varphi(e)$, for each $j \in N_1$ and $E_j^\varphi(e') \geq E_j^\varphi(e)$, for each $j \in N$, as desired.

Suppose there exists $i \in N$ such that $E_i^\varphi(e') < E_i^\varphi(e)$. Then, $\psi_i(E_i^\varphi(e')) < \psi_i(E_i^\varphi(e))$. Note that, in this case, $i \in N_1$. Then, $\psi_j(E_j^\varphi(e)) = \psi_i(E_i^\varphi(e)) > \psi_i(E_i^\varphi(e'))$, for each $j \in N_1$. Assume that $E_j^\varphi(e') > 0$ (otherwise, there is nothing to prove). Then, $\psi_j(E_j^\varphi(e)) > \psi_i(E_i^\varphi(e')) \geq \psi_j(E_j^\varphi(e'))$ and, therefore, $E_j^\varphi(e') < E_j^\varphi(e)$, as desired. To conclude, suppose, by contradiction, that there exists $j \in N \setminus N_1$ such that $E_j^\varphi(e') > E_j^\varphi(e) = 0$. Then, $\psi_j(E_j^\varphi(e)) \geq \psi_i(E_i^\varphi(e)) > \psi_i(E_i^\varphi(e')) = \psi_j(E_j^\varphi(e'))$, which contradicts the monotonicity of ψ_j.

We now show that if a rule R satisfies PR and SL then it is a member of the family $\{E^\varphi\}_{\varphi\in\Phi}$.

Let R be a rule satisfying PR and SL. By Theorem 1 in Moreno-Ternero and Roemer (2006) there is an index $\varphi : \mathbb{R}^2_{++} \cup \{(0,0)\} \to \mathbb{R}_+$, continuous on its domain and non-decreasing, such

[10]Note that each rule in $\{E^\varphi\}_{\varphi\in\Phi}$ is well defined thanks to the continuity and monotonicity properties of each $\varphi \in \Phi$.

that $\inf\{\varphi(x,y)\} = \varphi(0,0) = 0$ and for all $(x,y) > (z,t)$, $\varphi(x,y) > \varphi(z,t)$, for which[11]

$$R(e) = E^{\varphi}(e) \text{ for each } e \in \mathcal{E}^0.$$

Without loss of generality, let us denote by 1 the agent in $\mathbb{I}$ whose outcome function is defined by $u_1(x) = x$, for each $x \in \mathbb{R}_+$. The following claim, which holds thanks to PR and SL, will be used throughout the proof.

Claim. Let $e = (\{1,l\},(u_1,u_l),W) \in \mathcal{E}^0$ and $R(e) = (W - \hat{x}, \hat{x})$. Then, $R(e^j) = R(e)$ for each $e^j = (\{1,j\},(u_1,u_j),W) \in \mathcal{E}$, in which $j \in \mathbb{I}$ is such that $u_j(\hat{x}) = u_l(\hat{x})$.

Proof of the claim. Let $e = (\{1,l\},(u_1,u_l),W) \in \mathcal{E}^0$ and $R(e) = (W - \hat{x}, \hat{x})$. Let $j \in \mathbb{I}$ be such that $u_j(\hat{x}) = u_l(\hat{x})$, and $e^j = (\{1,j\},(u_1,u_j),W) \in \mathcal{E}$. Let $\hat{e} = (\{1,j,l\},(u_1,u_j,u_l),W + \hat{x}) \in \mathcal{E}$ and $\gamma = (\gamma_1,\gamma_2,\gamma_3) = R(\hat{e})$. If $\gamma_3 < \hat{x}$ then, by PR, $\gamma_2 < \hat{x}$.[12] By feasibility, $\gamma_1 > W - \hat{x}$, which contradicts SL, as $R(e) = (W - \hat{x}, \hat{x})$. Similarly, If $\gamma_3 > \hat{x}$ then, by PR, $\gamma_2 > \hat{x}$.[13] By feasibility, $\gamma_1 < W - \hat{x}$, which contradicts SL. Thus, $\gamma_3 = \hat{x}$. Then, by PR, $\gamma_2 = \hat{x}$, and, by feasibility, $\gamma_1 = W - \hat{x}$. Altogether, we have $R(\hat{e}) = (W - \hat{x}, \hat{x}, \hat{x})$. Finally, by CY, $R(e^j) = (W - \hat{x}, \hat{x}) = R(e)$, as desired.

For each $y \in \mathbb{R}_+$, let

$$\Omega^y = \{W \in \mathbb{R}_{++} : R_j(e) > 0, e = (\{1,j\},(u_1,u_j),W), j \in \mathbb{I}, u_j(0) = y\}.$$

In other words, Ω^y comprises the values of wealth for which the resulting two-agent economy involving 1 and an agent j with a strictly positive starting point is awarded a strictly positive amount. Note that the identity of j is irrelevant and only his starting point matters for the definition of Ω^y. More precisely, Ω^y remains the same for each agent $j \in \mathbb{I}$, such that $u_j(0) = y$.

We extend the definition of φ to include the set $\{0\} \times \mathbb{R}_{++}$ on its domain, as follows:

$$\varphi(0,y) = \inf \Omega^y \text{ for each } y \in \mathbb{R}_{++}.$$

We show first that $\varphi : \{\mathbb{R}_+ \times \mathbb{R}_{++}\} \cup \{(0,0)\} \to \mathbb{R}_+$, so extended, belongs to Φ. Then, to conclude the proof of the theorem, we show that $R(e) = E^{\varphi}(e)$ for each $e \in \mathcal{E} \setminus \mathcal{E}^0$.

[11] Without loss of generality, we can assume that $\varphi(x,x) = x$, for each $x \in \mathbb{R}_{++}$. This is due to the fact that, as shown in the proof of Theorem 1 in Moreno-Ternero and Roemer (2006), for each $(x,y) \in \mathbb{R}^2_{++}$, $\varphi(x,y)$ is defined as the unique value α for which an agent achieves the resource-outcome pair (x,y) in an economy granting α to agent 1. Thus, as $u_1(x) = x$, for each $x \in \mathbb{R}_{++}$, it follows that $\varphi(x,x) = x$, for each $x \in \mathbb{R}_{++}$.

[12] Otherwise, $\gamma_3 < \hat{x} \leq \gamma_2$ and therefore $u_l(\gamma_3) < u_l(\hat{x}) = u_j(\hat{x}) \leq u_j(\gamma_2)$, a contradiction with PR.

[13] Otherwise, $\gamma_3 > \hat{x} \geq \gamma_2$ and therefore $u_l(\gamma_3) > u_l(\hat{x}) = u_j(\hat{x}) \geq u_j(\gamma_2)$, a contradiction with PR.

- φ exhibits the following monotonicity properties:

  (*i*) If $x \in \mathbb{R}_+$ and $x', y \in \mathbb{R}_{++}$ are such that $x < x'$ then $\varphi(x,y) \leq \varphi(x',y)$.

  (*ii*) If $x \in \mathbb{R}_+$ and $y, y' \in \mathbb{R}_{++}$ are such that $y < y'$ then $\varphi(x,y) \leq \varphi(x,y')$.

  (*iii*) If $x \in \mathbb{R}_+$ and $x', y, y' \in \mathbb{R}_{++}$ are such that $(x,y) < (x',y')$, then $\varphi(x,y) < \varphi(x',y')$.

  (*iv*) $\varphi(0,0) = 0 \leq \varphi(x,y)$ for all $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_{++}$.

Given the monotonicity properties of φ on $\mathbb{R}^2_{++}$ described above, it suffices to prove the above items when $x = 0$ in each of them.

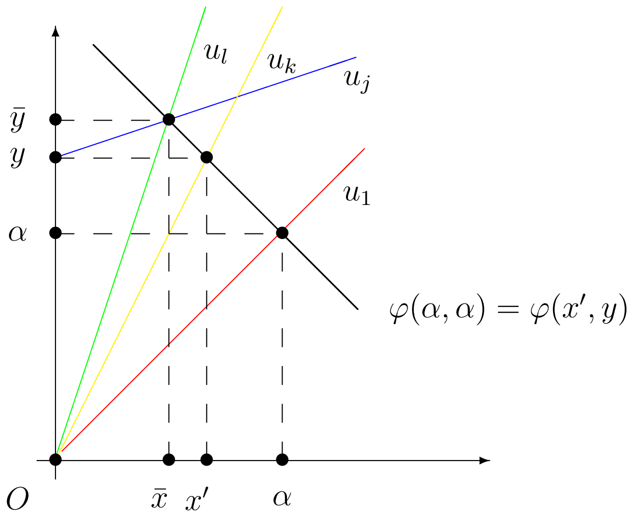

Illustration of the proof of statement (i)

Let $x', y \in \mathbb{R}_{++}$ and $\alpha = \varphi(x',y)$. Assume, without loss of generality, that $x' < y$.[14] Let $k \in \mathbb{I}$ be an agent whose outcome function satisfies that $u_k(0) = 0$ and $u_k(x') = y$. Then, consider the economy $e_k = (\{1,k\},(u_1,u_k),\alpha + x')$. As $e_k \in \mathcal{E}^0$, it follows that $R(e_k) = E^{\varphi}(e_k) = (\alpha, x')$. By PR, $x' \leq \alpha \leq y$.[15] Let $j \in \mathbb{I}$ be an agent whose outcome function satisfies that $u_j(0) = y$. As the iso-level sets of the index φ on $\mathbb{R}^2_{++}$ slope downward to the right, there exists $(\bar{x},\bar{y}) \in \mathbb{R}^2_{++}$, with

[14] Once we show this case, the alternative one (i.e., $x' \geq y$) would follow from this one and the monotonicity properties of φ on $\mathbb{R}^2_{++}$. Alternatively, a direct proof for that case could be provided making further use of PR. More precisely, if $x' > y$ then, by PR, $x' \geq \alpha \geq y$. Let $e_j^* = (\{1,j\},(u_1,u_j),\alpha)$. If $R_j(e_j^*) > 0$ there is nothing to prove, as $\alpha = \varphi(x',y) \in \Omega^y$ and, therefore, $\varphi(0,y) = \inf \Omega^y \leq \varphi(x',y)$, as desired. On the other hand, if $R_j(e_j^*) = 0$, let $\varepsilon > 0$ be given and consider the economy $e_j^{\varepsilon} = (\{1,j\},(u_1,u_j),\alpha+\varepsilon)$. If, by contradiction, $R_j(e_j^{\varepsilon}) = 0$, then, by PR, $y = u_j(0) \geq u_1(\alpha+\varepsilon) = \alpha+\varepsilon > y$, a contradiction. Thus, $\alpha + \varepsilon \in \Omega^y$ for each $\varepsilon > 0$, which implies that $\varphi(0,y) = \alpha = \inf \Omega^y \leq \varphi(x',y)$, as desired.

[15] Otherwise, if $\alpha < x'$, then, by PR, $u_1(\alpha) = \alpha \geq u_k(x') = y > x'$, a contradiction. Similarly, if $\alpha > y > x'$, then, by PR, $u_1(\alpha) = \alpha \leq u_k(x') = y$, a contradiction.

$x' \leq \bar{x}$ and $y \geq \bar{y}$, such that $\varphi(\bar{x}, \bar{y}) = \varphi(x', y) = \alpha = u_j(\bar{x})$. Assume, without loss of generality, that $(\bar{x}, \bar{y}) \neq (0, y)$.[16] Let $l \in \mathbb{I}$ be an agent whose outcome function satisfies that $u_l(0) = 0$ and $u_l(\bar{x}) = u_j(\bar{x})$ and consider the economy $e_l = (\{1, l\}, (u_1, u_l), \alpha + \bar{x})$. As $e_l \in \mathcal{E}^0$, it follows that $R(e_l) = E^{\varphi}(e_l) = (\alpha, \bar{x})$. Finally, consider the economy $e_j = (\{1, j\}, (u_1, u_j), \alpha + \bar{x})$. By the above claim, $R(e_j) = R(e_l) = (\alpha, \bar{x})$. By RM, $R_j(e_j^*) > 0$, where $e_j^* = (\{1, j\}, (u_1, u_j), \alpha)$. Thus, $\alpha = \varphi(x', y) \in \Omega^y$ and, therefore, $\varphi(0, y) = \inf \Omega^y \leq \varphi(x', y)$, as desired.

As for statement (ii), let $y, y' \in \mathbb{R}_{++}$ be such that $y < y'$. We show next that $\Omega^{y'} \subset \Omega^y$, from where it would follow that $\varphi(0, y) = \inf \Omega^y \leq \inf \Omega^{y'} = \varphi(0, y')$, as desired. Let $W \in \Omega^{y'}$. Then, there exists $k \in \mathbb{I}$ such that $u_k(0) = y'$ and for which $R_k(e_k) > 0$, where $e_k = (\{1, k\}, (u_1, u_k), W)$. Now, let $j \in \mathbb{I}$ be such that $u_j(0) = y$ and consider the economy $e_j = (\{1, j\}, (u_1, u_j), W)$. It suffices to show that $R_j(e_j) > 0$, as that would guarantee that $\varphi(0, y) \leq W$. Suppose otherwise, and let $e = (\{1, j, k\}, (u_1, u_j, u_k), W)$. Then, by SL applied to e_j and e, it follows that $R_j(e) = 0$. Then, by PR applied to e, it follows that $R_k(e) = 0$ and, therefore, $R(e) = (W, 0, 0)$. Finally, by SL applied to e_k and e, it follows that $R_k(e) = 0$, a contradiction.

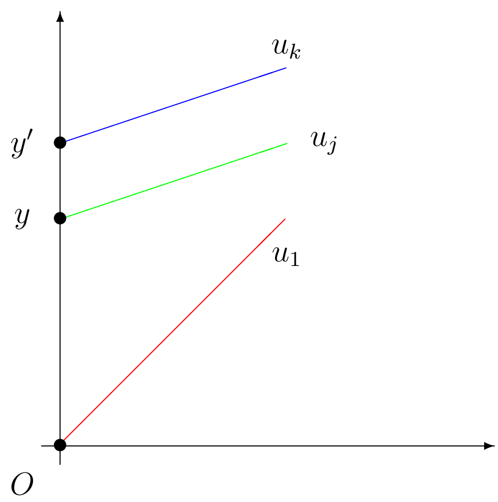

Illustration of the proof of statement (ii)

As for statement (iii), let $x', y, y' \in \mathbb{R}_{++}$ be such that $y < y'$. Then, by statement (i), $\varphi(0, y) \leq \varphi(x'/2, y)$. By the monotonicity properties of φ on $\mathbb{R}^2_{++}$, $\varphi(x'/2, y) \leq \varphi(x', y')$. Thus, $\varphi(0, y) < \varphi(x', y')$, as desired.

Finally, statement (iv) follows from the definition of Ω^y.

[16] Otherwise, the φ-level curve would be horizontal from $(0, y)$ to (x', y). But if so we could modify the ensuing argument to show that $R_j(\{1, j\}, (u_1, u_j), \alpha + \varepsilon) > 0$, for each $\varepsilon > 0$, which would also guarantee that $\alpha = \varphi(x', y) \in \Omega^y$ and, therefore, $\varphi(0, y) = \inf \Omega^y \leq \varphi(x', y)$, as desired.

- φ exhibits the following continuity properties.

  (i) $\lim_{(x,y)\to(x_0,y_0)} \varphi(x,y) = \varphi(x_0,y_0)$, for all $(x_0,y_0) \in \mathbb{R}^2_{++}$.

  (ii) $\lim_{x\to 0^+} \varphi(x,y(x)) = \varphi(0,y_0)$, for each $y : \mathbb{R}_{++} \to \mathbb{R}_{++}$ continuous and increasing function such that $\lim_{x\to 0^+} y(x) = y_0$.[17]

Only statement (ii) remains to be proved. Let $y_0 \in \mathbb{R}_{++}$ and $\{x_n\} \to 0^+$, with $x_n \in \mathbb{R}_{++}$. Let $y : \mathbb{R}_{++} \to \mathbb{R}_{++}$ be a continuous and increasing function such that $\lim_{x\to 0^+} y(x) = y_0$. For each n, we denote $y_n = y(x_n)$. Our aim is to show that $\{\alpha_n\} = \{\varphi(x_n,y_n)\} \to \alpha = \varphi(0,y_0)$. If such is not the case, then there exists a subsequence $\{\alpha_{k_n}\}$ that converges to $\overline{\alpha} \neq \alpha$.

Assume first that $\overline{\alpha} < \alpha$. Let B be a ball around $(0,y_0)$ and $\widehat{B} = \{(x,y) \in B : x > 0, y > y_0\}$. As $(x_n,y_n) \to (0,y_0)$, it follows that, for large k_n, $(x_{k_n},y_{k_n}) \in \widehat{B} \cup \{(0,y_0)\}$. Thus, by the monotonicity properties of φ described above, $\varphi(x_{k_n},y_{k_n}) > \alpha$.[18] On the other hand, as $\alpha_{k_n} \to \overline{\alpha} < \alpha$, it follows that, for large k_n, $\varphi(x_{k_n},y_{k_n}) = \alpha_{k_n} < \alpha$, which represents a contradiction.

Assume now that $\overline{\alpha} > \alpha$. Then, $\overline{\alpha} > \frac{\overline{\alpha}+\alpha}{2} > \alpha$. By the monotonicity properties of φ described above, there exists a ball B around $(0,y_0)$ such that $\varphi(x,y) < \frac{\overline{\alpha}+\alpha}{2}$, for each $(x,y) \in \widehat{B} = \{(x,y) \in B : x > 0, y > y_0\}$.[19] As $(x_n,y_n) \to (0,y_0)$, it follows that, for large k_n, $(x_{k_n},y_{k_n}) \in \widehat{B}$ and, therefore, $\varphi(x_{k_n},y_{k_n}) < \frac{\overline{\alpha}+\alpha}{2}$. On the other hand, as $\alpha_{k_n} \to \overline{\alpha} > \alpha$, it follows that, for large k_n, $\varphi(x_{k_n},y_{k_n}) = \alpha_{k_n} > \frac{\overline{\alpha}+\alpha}{2} > \alpha$, which represents a contradiction.

The above shows that $\varphi \in \Phi$. We now conclude the proof of the theorem by showing that $R(e) = E^{\varphi}(e)$ for each $e \in \mathcal{E} \setminus \mathcal{E}^0$. By the above claim, and the fact that R coincides with E^{φ} on $\mathcal{E}^0$, it follows that coincidence also holds for the domain of two-agent economies (in $\mathcal{E}$) involving agent 1.

More precisely, let $j \in \mathbb{I}$ be such that $u_j(0) > 0$, and $e = (\{1,j\},(u_i,u_j),W) \in \mathcal{E}$. In what follows, and for ease of notation, we denote agent j by 2. We claim that $x = (x_1,x_2) = R(e) =$

[17] Note, in particular, that $y(x) \geq y_0$, for each $x \in \mathbb{R}_{++}$.

[18] More generally, it actually holds that $\widehat{B}$ does not contain any point in the $R-$iso-level set for another value $\widehat{\alpha} < \alpha$. Assume otherwise. Then, let $(x,y) \in B$ such that $x > 0$ and $y > y_0$ and for which there exists $j \in \mathbb{I}$, with $u_j(0) = 0$ and $u_j(x) = y$, and $\widehat{\alpha} < \alpha$, such that $R(e) = (\widehat{\alpha}, x)$, where $e = (\{1,j\},(u_1,u_j),\widehat{\alpha}+x)$. As $e \in \mathcal{E}^0$, it follows that $R(e) = E^{\varphi}(e)$. Thus, $\varphi(x,y) = \varphi(\widehat{\alpha},\widehat{\alpha}) = \widehat{\alpha} < \alpha$. On the other hand, by the monotonicity properties of φ, we obtain that $\varphi(x,y) > \varphi(0,y_0) = \alpha$, a contradiction.

[19] More precisely, as $\varphi(0,y_0) = \alpha < \frac{\overline{\alpha}+\alpha}{2} < \overline{\alpha}$, it follows that there exists $(x,y) \in \mathbb{R}^2_{++}$ with $y > y_0$, such that $\varphi(x,y) = \frac{\overline{\alpha}+\alpha}{2}$. Then, we can construct a ball B around $(0,y_0)$ such that $\widehat{B}$ lies below the $\varphi-$iso-level set corresponding to $\frac{\overline{\alpha}+\alpha}{2}$.

$E^{\varphi}(e) = y = (y_1, y_2)$.[20]

Assume first that $x_2 > 0$. Then, suppose, without loss of generality, that $3 \in \mathbb{I}$, is such that $u_3(0) = 0$ and $u_3(x_2) = u_2(x_2)$. Let $z = (z_1, z_2, z_3) = R(\{1, 2, 3\}, (u_1, u_2, u_3), W + x_2)$. If $z_3 < x_2$, then, by PR, $z_2 < x_2$.[21] By feasibility, $z_1 > x_1$. This violates SL. If $z_3 > x_2$, then, by PR, $z_2 > x_2$. By feasibility, $z_1 = x_1$. This violates SL. Then, $z_3 = x_2$. By PR, $z_2 = x_2$. By feasibility $z_1 < x_1$. By SL, $R(\{1, 3\}, (u_1, u_3), W) = (x_1, x_2)$. As $(\{1, 3\}, (u_1, u_3), W) \in \mathcal{E}^0$, then $\varphi(x_1, x_1) = \varphi(x_2, u_3(x_2)) = \varphi(x_2, u_2(x_2))$ and $(x_1, x_2) = E^{\varphi}(\{1, 3\}, (u_1, u_3), W)$, as desired.

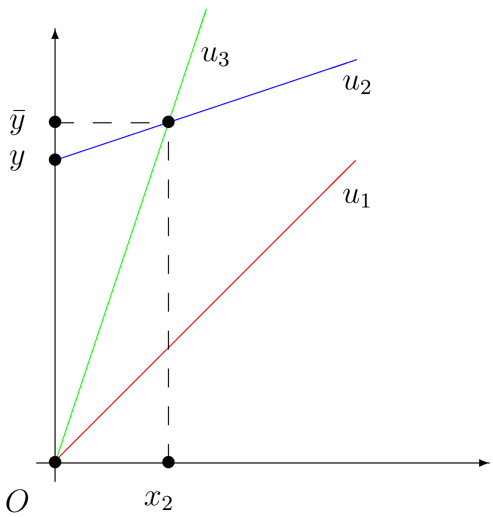

Illustration of the above proof

Assume now that $x_2 = 0$. Suppose, by contradiction, that $y_2 > 0$. Then, by definition of E^{φ}, it follows that $y_1 = \varphi(y_1, y_1) = \varphi(y_2, u_2(y_2))$. By the monotonicity properties of φ, $\varphi(y_2, u_2(y_2)) > \varphi(0, u_2(0))$. Now, if $x_2 = 0$, it follows, by the definition of Ω^y, that $\varphi(0, u_2(0)) \geq W$. Altogether, we obtain that $y_1 > W$, a contradiction.

The above shows that R coincides with E^{φ} on the domain of two-agent economies involving agent 1. To conclude, let us consider a general economy $e = (N, u, W) \in \mathcal{E}$.

By RM and PR, there is $W' > W$ such that $R_1(e') = W' - W$, where $e' = (\{1\} \cup N, (u_1, u), W')$.

Formally, let $\Omega^{<} = \{W' \in (W, +\infty) : R_1(e') < W' - W\}$ and $\Omega^{>} = \{W \in (W, +\infty) : R_1(e') > W' - W\}$.

[20]Note that if $u_2(0) = 0$, there is nothing to prove, thanks to Theorem 1 in Moreno-Ternero and Roemer (2006). Furthermore, note that, if $x_1 < y_1$ then, by PR, $x_1 > 0$. More precisely, if $x_1 = 0$ then $x_2 = W > 0$ and, by PR, $u_2(W) \leq u_1(0) = 0$, which contradicts the fact that u_2 is strictly increasing. More generally, the previous argument shows that $R_i(e) > 0$, for each $i \in N$ such that $u_i(0) = 0$.

[21]Otherwise, $u_2(z_2) \geq u_2(x_2) > u_2(z_3)$ and $z_2 \geq x_2 > z_3$, a contradiction.

We show first that $\Omega^{<} \neq \emptyset \neq \Omega^{>}$.

More precisely, let $j \in N$ be such that $R_j(e) > 0$ and ε be such that $0 < \varepsilon < \min\{R_j(e), u_j(R_j(e))\}$. Let $W' = W + \varepsilon$ and assume, by contradiction, that $W' \notin \Omega^{>}$, i.e., $R_1(e') = R_1(\{1\} \cup N, (u_1, u), W + \varepsilon) \leq \varepsilon$. Then, by feasibility, $\sum_{k \in N} R_k(e') \geq W$. By SL, $R_k(e') \geq R_k(e)$, for each $k \in N$. Thus, $R_j(e') \geq R_j(e) > \varepsilon \geq R_1(e')$, and $u_j(R_j(e')) \geq u_j(R_j(e)) > \varepsilon \geq R_1(e') = u_1(R_1(e'))$, which contradicts SL. Therefore, $W' \in \Omega^{>}$.

Similarly, let $k_0 = \arg\min_{k \in N} u_k(W)$ and $n \in \mathbb{N}$ be such that $(n-1)W > u_{k_0}(W)$. Let $W' = nW$ and assume, by contradiction, that $W' \notin \Omega^{<}$, i.e., $R_1(e') = R_1(\{1\} \cup N, (u_1, u), nW) \geq (n-1)W$. Then, by feasibility, $\sum_{k \in N} R_k(e') \leq W$. By SL, $R_k(e') \leq R_k(e)$, for each $k \in N$. Thus, $R_{k_0}(e') \leq W < (n-1)W \leq R_1(e')$, and $u_{k_0}(R_{k_0}(e')) \leq u_{k_0}(W) < (n-1)W \leq R_1(e') = u_1(R_1(e'))$, which contradicts SL. Therefore, $W' \in \Omega^{<}$.

It is obvious that $\Omega^{<} \cap \Omega^{>} = \emptyset$. We show now that both are open sets.

Let $W' \in \Omega^{<}$ and denote $\alpha' = R_1(e') < W - W'$. Let $\varepsilon = \frac{W' - W - \alpha'}{2}$. By RM, $(W' - \varepsilon, W'] \subset \Omega^{<}$. Suppose, by contradiction, that there exists $W^* \in (W', W' + \varepsilon)$ such that $W^* \notin \Omega^{<}$, i.e., $R_1(e^*) \geq W' - W$, for $e^* = (\{1\} \cup N, (u_1, u), W^*)$. By RM, $\sum_N R_j(e^*) \geq \sum_N R_j(e') = W' - \alpha'$. Then, $W^* \geq W' - W + W' - \alpha'$, which contradicts that $W^* \in (W', W' + \varepsilon)$. This shows that $\Omega^{<}$ is an open set. Analogously, we show that $\Omega^{>}$ is an open set.

If, by contradiction, we assume that $R_1(e') \neq W' - W$, for each $W' \in (W, +\infty)$, then $(W, +\infty) = \Omega^{>} \cup \Omega^{<}$. It would then follow that $(W, +\infty)$ is not connected, which is a contradiction.

Let $x = R_N(e')$ and, for each $j \in N$, $e_j = (\{1, j\}, (u_1, u_j), W' - W + x_j)$. By CY, $R(e_j) = R_{\{1,j\}}(e') = (W' - W, x_j)$. Now, as e_j is a two-agent economy involving agent 1, it also follows that $R(e_j) = E^{\varphi}(e_j)$. Thus, for each $j \in N$, $R(e_j) = E^{\varphi}(e_j)$, and, therefore, by SL, $E^{\varphi}(e') = R(e')$.[22] Finally, by CY, $E^{\varphi}(e) = x = R(e)$, as desired.

[22] More precisely, suppose that $E_1^{\varphi}(e') > W' - W$. Then, by SL applied to e_j and e', $E_j^{\varphi}(e') > x_j$, for each $j \in N$, which represents a contradiction with feasibility. We reach a similar contradiction assuming that $E_1^{\varphi}(e') < W' - W$. Thus, $E_1^{\varphi}(e') = W' - W = E_1^{\varphi}(e_j)$. Now, if agent 1 does not change moving from e_j to e', and gets a positive amount in each case, it follows that no other agent can get different amounts.

8 Proof of Theorem 2

By Theorem 1, we know that RE and COE satisfy PR and SL. It is straightforward to show that RE satisfies CP.[23] We then show that COE satisfies CP.

Let $e = (N, u, W) \in \mathcal{E}$. For ease of exposition, assume that $N = \{1, 2, ..., n\}$ and that agents are ranked (in an increasing order) according to their initial starting points, i.e., $u_i(0) \leq u_{i+1}(0)$ for each $i = 1, ..., n-1$. Let $W^1, W^2 \in \mathbb{R}_{++}$ be such that $W = W^1 + W^2$ and let $e^1 = (N, u, W^1) \in \mathcal{E}$. For each $i \in N$, let $\hat{i} \in \mathbb{I}$ be such that $\widehat{u}_{\hat{i}}(x) = u_i(x + COE_i(e^1))$ for each $x \in \mathbb{R}_+$, and let $e^2 = (\widehat{N}, \widehat{u}, W^2) \in \mathcal{E}$, where $\widehat{u} = (\widehat{u}_{\hat{i}})_{i \in N}$. Let $\sigma_i = u_i^{-1}$ and $\widehat{\sigma}_{\hat{i}} = \widehat{u}_{\hat{i}}^{-1}$ for each $i \in N$. Then,

$$COE_i(e) = \begin{cases} \sigma_i(\lambda) & \text{for each } i = 1, \ldots, k \\ 0 & \text{for each } i = k+1, \ldots, n \end{cases}$$

where λ and k are such that

$$\sum_{i=1}^{k} \sigma_i(\lambda) = W, \text{ and } u_{k+1}(0) > \lambda \geq u_k(0).$$

Similarly,

$$COE_i(e^1) = \begin{cases} \sigma_i(\lambda_1) & \text{for each } i = 1, \ldots, k_1 \\ 0 & \text{for each } i = k_1+1, \ldots, n \end{cases}$$

where λ_1 and k_1 are such that

$$\sum_{i=1}^{k_1} \sigma_i(\lambda_1) = W^1, \text{ and } u_{k_1+1}(0) > \lambda_1 \geq u_{k_1}(0).$$

Thus, note that $k \geq k_1$ and $\lambda \geq \lambda_1$. Finally,

$$COE_i(e^2) = \begin{cases} \widehat{\sigma}_{\hat{i}}(\lambda_2) & \text{for each } i = 1, \ldots, k_2 \\ 0 & \text{for each } i = k_2+1, \ldots, n \end{cases}$$

where λ_2 and k_2 are such that

$$\sum_{i=1}^{k_2} \widehat{\sigma}_{\hat{k}_2}(\lambda_2) = W^2, \text{ and } \widehat{u}_{\widehat{k_2+1}}(0) > \lambda_2 \geq \widehat{u}_{\hat{k}_2}(0).$$

Let $y = COE(e) - COE(e^1)$ and $z = COE(e^2)$. We have to show that $y = z$. To do so, note first that

$$\widehat{u}_{\hat{i}}(y_i) = \begin{cases} \lambda & \text{for each } i = 1, \ldots, k \\ u_i(0) & \text{for each } i = k+1, \ldots, n \end{cases}$$

[23] Formally, let $e = (N, u, W) \in \mathcal{E}$ and $n = |N|$. Let $W^1, W^2 \in \mathbb{R}_{++}$ be such that $W = W^1 + W^2$ and let $e^1 = (N, u, W^1) \in \mathcal{E}$. For each $i \in N$, let $\hat{i} \in \mathbb{I}$ be such that $\widehat{u}_{\hat{i}}(x) = u_i(x + RE_i(e^1)) = u_i(x + \frac{W^1}{n})$ for each $x \in \mathbb{R}_+$, and let $e^2 = (\widehat{N}, \widehat{u}, W^2) \in \mathcal{E}$, where $\widehat{u} = (\widehat{u}_{\hat{i}})_{i \in N}$. Then, for each $i \in N$, $RE_i(e) = \frac{W}{n} = \frac{W^1}{n} + \frac{W^2}{n} = RE_i(e^1) + RE_i(e^2)$, as desired.

and

$$\widehat{u}_{\hat{i}}(z_i) = \begin{cases} \lambda_2 & \text{for each } i = 1, \ldots, k_2 \\ \widehat{u}_{\hat{i}}(0) & \text{for each } i = k_2 + 1, \ldots, n \end{cases}$$

Note also that, as y is a feasible allocation for the economy e^2, it follows, by definition of COE, that $(\widehat{u}_{\hat{i}}(z_i))_{i \in N}$ lexicographically dominates $(\widehat{u}_{\hat{i}}(y_i))_{i \in N}$. This implies that $\lambda \leq \lambda_2$. Thus, $k_1 \leq k \leq k_2$. Then,

$$y_i = \begin{cases} \sigma_i(\lambda) - \sigma_i(\lambda_1) & \text{for each } i = 1, \ldots, k_1 \\ \sigma_i(\lambda) & \text{for each } i = k_1 + 1, \ldots, k \\ 0 & \text{for each } i = k + 1, \ldots, n \end{cases}$$

and

$$z_i = \begin{cases} \widehat{\sigma}_{\hat{i}}(\lambda_2) & \text{for each } i = 1, \ldots, k_2 \\ 0 & \text{for each } i = k_2 + 1, \ldots, n \end{cases}$$

Let $i = 1, \ldots, k_1$. Then, $\widehat{u}_{\hat{i}}(x) = u_i(x + \sigma_i(\lambda_1))$ for each $x \in \mathbb{R}_+$. Thus, $\widehat{\sigma}_{\hat{i}}(x) = \sigma_i(x) - \sigma_i(\lambda_1)$ for each $x \in \mathbb{R}_+$. In particular, $\widehat{\sigma}_{\hat{i}}(\lambda_2) = \sigma_i(\lambda_2) - \sigma_i(\lambda_1)$. Similarly, $\widehat{\sigma}_{\hat{i}}(\lambda_2) = \sigma_i(\lambda_2)$ for each $i = k_1 + 1, \ldots, k_2$. Thus, $z_i \geq y_i$ for each $i \in N$. Now, if $\lambda < \lambda_2$, we would have $W^2 = \sum_{i \in N} z_i > \sum_{i \in N} y_i = W^2$, a contradiction. Thus, it follows that $\lambda = \lambda_2$ and, therefore, that $k = k_2$, which implies that $y = z$, as desired.

We conclude by showing that no other rule within the family $\{E^\varphi\}_{\varphi \in \Phi}$ satisfies CP.

Let $\widehat{\Phi}$ denote the residual of Φ after removing the functions giving rise to RE and COE. We partition the family $\widehat{\Phi}$ according to the following concept. We say that $\varphi \in \widehat{\Phi}$ is *quasilinear in x* if there exists $\lambda > 0$ and $f : \mathbb{R}_+ \to \mathbb{R}_+$, continuous and increasing, with $f(0) = 0$, and $f(x) > 0$ for some $x > 0$, such that $\varphi(x, y) = \lambda x + f(y)$, for each $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_{++}$.[24]

Case 1. The quasiliniear case.

Let $\varphi \in \widehat{\Phi}$ be a quasilinear function in x. Then, there exists $y_2 \in \mathbb{R}_{++}$ such that $\varphi(0, y_2) > 0$. Furthermore, there exist $\delta \in \mathbb{R}_{++}$ and $(x_1, y_1) \in \mathbb{R}^2_{++}$, with $x_1 > \delta$ and $y_2 > y_1$, such that $\varphi(x_1 - \delta, y_1) = \varphi(0, y_2) > 0$.[25]

[24] In other words, the level curves of φ are parallel displacements of each other along the x axis. Note that RE and COE emerge from degenerate quasilinear functions. More precisely, if $\lambda = 0$, and f is strictly increasing, the corresponding E^φ rule would be COE, whereas if $f(x) = 0$ for each $x \in \mathbb{R}_+$, and $\lambda > 0$, then the corresponding E^φ rule would be RE.

[25] Note that $\varphi(x_1 - \delta, y_1) = \lambda(x_1 - \delta) + f(y_1)$, whereas $\varphi(0, y_2) = f(y_2) > 0$. Thus, as $f(0) = 0$, the equality is guaranteed by the continuity and monotonicity of f.

Let $u_1, u_2 \in \mathcal{U}$ be such that

$$u_1(0) = 0, \quad u_2(0) = y_2, \quad u_1(x_1) = y_1.^{26}$$

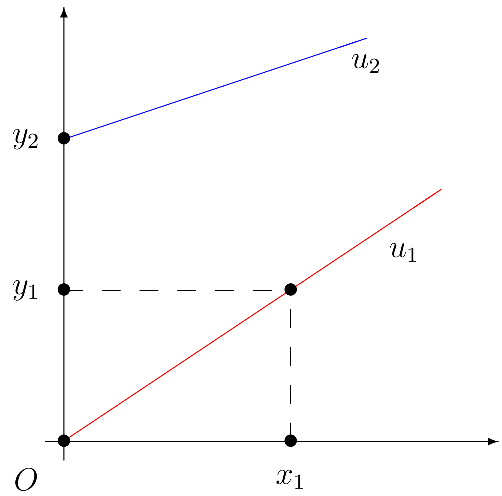

Illustration of the proof of Case 1

Let $\lambda = \psi_1(x_1) = \varphi(x_1, y_1) > 0$ and $W = \psi_1^{-1}(\lambda) + \psi_2^{-1}(\lambda)$ and consider the economy $e = (\{1, 2\}, (u_1, u_2), W)$.[27] Then, it is straightforward to show that

$$E^{\varphi}(e) = \left(\psi_1^{-1}(\lambda), \psi_2^{-1}(\lambda)\right).$$

Let $\varepsilon \in \mathbb{R}_{++}$ be such that $\varepsilon < W$ and $\varphi(\varepsilon, u_1(\varepsilon)) < \varphi(0, y_2)$ and consider the economy $e^1 = (\{1, 2\}, (u_1, u_2), \varepsilon)$.[28] Then, it is straightforward to show that

$$E^{\varphi}(e^1) = (\varepsilon, 0).$$

Finally, let $W_2 = W - \varepsilon > 0$. For $i = 1, 2$, let $\widehat{u}_{\hat{i}} : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $\widehat{u}_{\hat{i}}(x) = u_i(x + E_i^{\varphi}(e^1))$ and consider the economy $e^2 = (\{\widehat{1}, \widehat{2}\}, (\widehat{u}_{\hat{1}}, \widehat{u}_{\hat{2}}), W^2)$. For $i = 1, 2$, let $\hat{\psi}_i : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $\hat{\psi}_i(w) = \varphi(w, \widehat{u}_{\hat{i}}(w))$ for each $w \in \mathbb{R}_+$.

Assume, by contradiction, that E^{φ} satisfies CP. Thus, $E^{\varphi}(e^2) = E^{\varphi}(e) - E^{\varphi}(e^1) = \left(\psi_1^{-1}(\lambda) - \varepsilon, \psi_2^{-1}(\lambda)\right)$. Then, if $E^{\varphi}(e^2) = (W_2, 0)$, it follows that $\psi_2^{-1}(\lambda) = 0$, which would

[26] Consequently, $u_1(x) < u_2(x)$ for each $x \leq x_1$.

[27] Note that $\psi_2(0) = \varphi(0, y_2) = \varphi(x_1 - \delta, y_1) < \varphi(x_1, y_1) = \lambda$, where the last inequality follows from the fact that φ is quasilinear in x. Thus, $\psi_2^{-1}(\lambda) > 0$, and e is well defined. Furthermore, the solution that the rule E^{φ} yields for such economy is indeed obtained upon equalizing the index φ.

[28] As before, the existence of ε is guaranteed by the continuity and monotonicity of f. Furthermore, the solution that the rule E^{φ} yields for such economy cannot be obtained upon equalizing the index φ and thus, agent 2 gets nothing.

imply $\varphi(x_1, y_1) = \lambda = \varphi(0, y_2) = \varphi(x_1 - \delta, y_1)$, a contradiction with the fact that φ is quasilinear. If, on the other hand, $E^{\varphi}(e^2) = \left(\hat{\psi}_1^{-1}(\lambda'), \hat{\psi}_2^{-1}(\lambda')\right)$, then $\psi_2^{-1}(\lambda) = \hat{\psi}_2^{-1}(\lambda')$ and $x_1 = \psi_1^{-1}(\lambda) = \varepsilon + \hat{\psi}_1^{-1}(\lambda')$. From the former equality, it follows that $\lambda = \lambda'$, as $\hat{\psi}_2 \equiv \psi_2$ is a strictly increasing function. Thus, from the latter equality, it follows that $x_1 = \varepsilon + \hat{\psi}_1^{-1}(\lambda)$, or, equivalently, $\varphi(x_1, y_1) = \lambda = \hat{\psi}_1(x_1 - \varepsilon) = \varphi(x_1 - \varepsilon, \widehat{u}_{\hat{1}}(x_1 - \varepsilon)) = \varphi(x_1 - \varepsilon, u_1(x_1)) = \varphi(x_1 - \varepsilon, y_1)$, again, a contradiction with the fact that φ is quasilinear.

Case 2. The non-quasiliniear case.

Let $\varphi \in \widehat{\Phi}$ be a non-quasilinear function in x. Then, there exist $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2_{++}$, and $0 < \alpha < \min\{x_1, x_2\}$, such that $\varphi(x_1, y_1) = \varphi(x_2, y_2)$ and $\varphi(x_1 - \alpha, y_1) \neq \varphi(x_2 - \alpha, y_2)$.[29]

Let $u_1, u_2 \in \mathcal{U}$ be such that

$$u_1(x_1) = y_1, \quad u_2(x_2) = y_2, \quad u_1(\alpha) = u_2(\alpha).$$

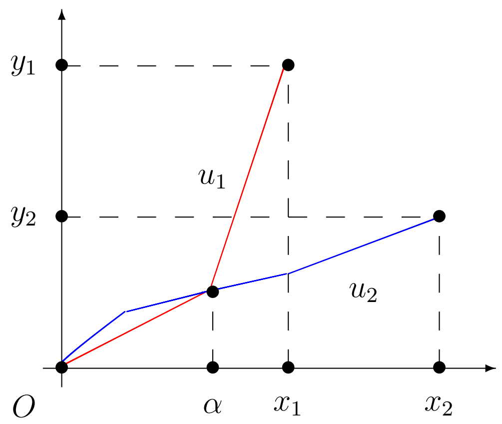

Illustration of the proof of Case 2

Now, consider the economies $e^1 = (\{1, 2\}, (u_1, u_2), 2\alpha)$ and $e = (\{1, 2\}, (u_1, u_2), x_1 + x_2)$. As $u_1(\alpha) = u_2(\alpha)$ and $\varphi(x_1, u_1(x_1)) = \varphi(x_2, u_2(x_2))$, it follows that

$$E^{\varphi}(e^1) = (\alpha, \alpha),$$

and

$$E^{\varphi}(e) = (x_1, x_2)\,.$$

For $i = 1, 2$, let $\widehat{u}_{\hat{i}} : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $\widehat{u}_{\hat{i}}(x) = u_i(x + \alpha)$ and consider the economy $e^2 = (\{\widehat{1}, \widehat{2}\}, (\widehat{u}_{\hat{1}}, \widehat{u}_{\hat{2}}), W^2)$, where $W^2 = x_1 + x_2 - 2\alpha$. For $i = 1, 2$, let $\hat{\psi}_i : \mathbb{R}_+ \to \mathbb{R}_+$ be such

[29] This is simply a consequence of the fact that, in this case, not all level curves of φ are parallel displacements of each other along the x axis.

that $\hat{\psi}_i(w) = \varphi(w, \widehat{u}_{\hat{i}}(w))$ for each $w \in \mathbb{R}_+$. Then,

$$E^\varphi(e^2) = \left(\hat{\psi}_1^{-1}(\lambda'), \hat{\psi}_2^{-1}(\lambda')\right),$$

where λ' is such that $\psi_1^{-1}(\lambda') + \psi_2^{-1}(\lambda') = W^2$.[30] Equivalently,

$$E^\varphi(e^2) = (x, x_1 + x_2 - 2\alpha - x),$$

where

$$\hat{\psi}_1(x) = \hat{\psi}_2(x_1 + x_2 - 2\alpha - x),$$

i.e.,

$$\varphi(x, \widehat{u}_{\hat{1}}(x)) = \varphi(x_1 + x_2 - 2\alpha - x, \widehat{u}_{\hat{2}}(x_1 + x_2 - 2\alpha - x)).$$

Equivalently,

$$\varphi(x, u_1(x + \alpha)) = \varphi(x_1 + x_2 - 2\alpha - x, u_2(x_1 + x_2 - \alpha - x)). \tag{2}$$

Now, $E^\varphi(e) = E^\varphi(e^1) + E^\varphi(e^2)$ if and only if $\alpha + x = x_1$. But if so, (2) becomes

$$\varphi(x_1 - \alpha, y_1) = \varphi(x_2 - \alpha, y_2),$$

which represents a contradiction.

9 Auxiliary statement

A rule satisfies solidarity if and only if it satisfies consistency and resource monotonicity.

Proof. Let R be a rule satisfying SL. We show that R satisfies RM and CY.

Let $e = (N, u, W)$ and $e' = (N, u, W') \in \mathcal{E}$ be such that $W' < W$. Let $N_1 = \{i \in N : R_i(e) > 0\}$. As $\sum_{i \in N} R(e) = W > W' = \sum_{i \in N} R(e')$, it follows, by SL, that $R_i(e') \leq R_i(e)$, for each $i \in N$, and $R_i(e') < R_i(e)$, for each $i \in N_1$, which shows that R satisfies RM.

As for CY, let $e = (N, u, W) \in \mathcal{E}$ and $N' \subset N$ such that $e' = (N', u', W') \in \mathcal{E}$, where $W' = \sum_{i \in N'} R_i(e)$. Suppose, by contradiction, that there exists $j \in N'$ such that $R_j(e) \neq R_j(e')$. Then, by SL, either $W' = \sum_{i \in N'} R_i(e) < \sum_{i \in N'} R_i(e') = W'$, or $W' = \sum_{i \in N'} R_i(e) > \sum_{i \in N'} R_i(e') = W'$, a contradiction in any case.

Conversely, let R be a rule satisfying RM and CY. Let $e = (N, u, W) \in \mathcal{E}$ and $e' = (N', u', W') \in \mathcal{E}$, be such that $N \subseteq N'$. Let $N_1 = \{i \in N : R_i(e) > 0\}$. Consider the auxiliary economy $\widehat{e} = (N, u, \widehat{W}) \in \mathcal{E}$, where $\widehat{W} = \sum_{i \in N} R_i(e')$. Then, by CY, $R(\widehat{e}) = R_N(e')$.[31] By

[30] Note that this is also a consequence of the fact that $u_1(\alpha) = u_2(\alpha)$, as such equality guarantees that the φ-value of both agents can be equalized in the resulting economy.

[31] Note that the bilateral version of the consistency axiom, in which $|N'| = 2$, would not be enough.

RM, one of the following three possibilities happens:

$$R(\widehat{e}) = R(e),$$

$$R_i(\widehat{e}) \leq R_i(e),\ \textit{for each } i \in N,\ \textit{and } R_i(e') < R_i(e),\ \textit{for each } i \in N_1,$$

$$R_i(\widehat{e}) \geq R_i(e),\ \textit{for each } i \in N,\ \textit{and } R_i(e') > R_i(e),\ \textit{for each } i \in N_1.$$

Altogether, we have that R satisfies SL. ■