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Un axioma de nulidad para juegos en forma de función de partición

A nullity axiom for games in partition function form

Joss Sánchez-Pérez
Universidad Autónoma de San Luis Potosí (México)
https://orcid.org/0000-0003-3922-9788
joss.sanchez@uaslp.mx

Antelmo Lozano-Martínez
Universidad Autónoma de San Luis Potosí (México)
A352717@alumnos.uaslp.mx

RESUMEN

Inspirándose en los fundamentos de Shapley, Myerson (1977) propuso una solución única para juegos en forma de función de partición caracterizados por tres axiomas: linealidad, simetría y el axioma del portador. Esta investigación tiene como objetivo determinar si se puede llegar a una caracterización alternativa mediante la inclusión de un axioma de nulidad. Hasta donde sabemos, no existe tal caracterización en la literatura actual. Se presenta una propuesta para una definición de jugador nulo, revelando que la caracterización derivada no es única. En cambio, se presenta una familia de soluciones parametrizadas, demostrando la diversidad de reparticiones potenciales en este contexto.

PALABRAS CLAVE

Juegos en forma de función de partición, valor de Myerson, jugadores nulos, valor de Shapley.

ABSTRACT

Drawing inspiration from Shapley's foundations, Myerson (1977) proposed a unique solution for games in partition function form characterized by three axioms: linearity, symmetry, and the carrier axiom. This research aimed to determine if an alternative characterization can be reached by including a nullity axiom. As far as we know, there is not such characterization in the current literature. A proposal for a null player definition is introduced, revealing that the derived characterization is not unique. Instead, a family of parameterized solutions is presented, demonstrating the diversity of potential outcomes in this context.

KEYWORDS

Games in partition function form, Myerson value, null players, Shapley value.

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1. INTRODUCTION

A game in partition function form refers to a situation in game theory where the actions of one player (or a group of players) affect other players in a way that is not reflected in the payoff functions of those players. In this case, externalities can be positive or negative.

Externalities can be positive when the action of one player benefits other players in the game without compensating them. For example, in a network effects scenario, the more users a platform has, the more valuable it becomes to all users. Also they can be negative when the action of one player imposes costs on other players in the game without compensating them. An example could be pollution caused by a factory affecting the health and well-being of people living nearby.

Games with externalities (modeled as games in partition function form) are interesting because the optimal strategies for each player may not take into account the full impact of their decisions on others. This can lead to suboptimal outcomes from a societal perspective, where the total welfare of all players could be improved if players coordinated or cooperated differently.

For instance, in environmental economics, the presence of negative externalities (like pollution) often leads to inefficient outcomes because individual firms may not fully account for the costs imposed on society. Game theory helps analyze such situations and suggest ways to achieve better outcomes through policy interventions, cooperation mechanisms, or changes in incentives.

This kind of games were introduced formally by Lucas and Thrall (1963) and Myerson (1977) extends the idea of the Shapley value (Shapley, 1953) to situations where the value of coalitions is defined by a more complex function (partition function form), where interactions among coalition members affect the total value.

It has been shown that Myerson's value can be characterized by the axioms of linearity, symmetry, and carrier (similarly to Shapley's value), but it was not clear if a unique characterization could be found using the axioms of linearity, symmetry, nullity, and efficiency (as is the case with Shapley). This characterization, based on a "null player", is crucial for understanding how benefits are distributed in cooperation situations where players can form coalitions.

Other authors have proposed different definitions to characterize a player as null in partition function form games. These definitions vary in their conditions and approaches. For example, Pham Do and Norde (2007) defined a player as null if their contribution to any coalition is zero, and if by changing coalitions, the total wealth is not affected. Macho-Stadler et al., (2007) proposed a similar definition, but without the condition that the null player does not generate value by themselves. Hafalir (2007) introduced the concept of an efficient-covering null player and compares how what he calls an "efficient partition" behaves instead of the original partition, where he explicitly assumes that the remaining players in the coalition are individuals (individual players without coalitions). Bolger (1989) defined a player as null in the strong sense if their transfer between coalitions does not affect the wealth of any coalition. Skibski (2011) introduced the concept of a null player with constant marginality, which characterizes such a player as one whose contribution to wealth is consistent and balanced in all possible partitions of the player set.

In our work, we have proposed a definition of a null player, based on Myerson's carrier axiom, that focuses on the redistribution of wealth in the game when the null player is present or absent in a coalition. Our definition evaluates how a partition P of players changes when the null player is involved in a coalition and when they are not. Unlike other definitions, our proposal does not establish specific conditions on the value of the null player or their impact on other coalitions. We have analyzed how our definition compares with other proposals in the literature and highlighted the differences in approach and requirements to be considered a null player.

In particular, we explore the possibility of finding a unique solution that satisfies the axioms of linearity, symmetry, nullity and efficiency. We find that our notion of a null player generates an infinite variety of solutions, and that Myerson's value represents a specific case within this set.

The article is organized as follows. In the next section we focus on establishing the necessary basic concepts and notation for the further development of our work. In Sect. 3 we present a new proposal for the definition of a null player (and its corresponding nullity axiom), directly derived form the carrier definition presented in Myerson (1977). A characterization of all solutions satisfying the properties of linearity, symmetry, nullity and efficiency is presented in Sect. 4. In the same section we contrast our results with those presented in related literature.

2. BACKGROUND CONCEPTS AND NOTATION

Game theory relies on fundamental mathematical concepts and we will follow standard notation, conventions and mathematical language.

Let $N=\{1,\dots,n\}$ be a finite set of players. A *coalition* is any subset S of N and let $2^N=\{T|T\subseteq N,T\neq\emptyset\}$ be the set of all non-empty subsets of N.

Let $S^{-i}=S\backslash\{i\}$ and $S^{+i}=S\cup\{i\}$. The cardinality or size of a set S, denoted by $|S|$, is the number of elements in S. A set consisting of a single element is called a singleton; the set containing no elements is the empty set, denoted by $\emptyset$.

A partition of $S\subseteq N$, is a set $\{S_1,S_2,\dots,S_m\}$ of subsets of S such that

$$\bigcup_{i=1}^{m} S_i = S, \quad S_j \cap S_k = \emptyset \quad \forall j \neq k$$

The set of partitions of S is denoted by $\tilde{\Pi}(S)$.

By convention, $\emptyset\in P$ for every partition P and $|P|$ will denote the number of nonempty sets in P. The partitioning of the set S into singletons is denoted by $[S]=\{\{j\}|j\in S\}$.

We say that a coalition C is embedded in partition P if $C\in P$; by embedded coalition we mean a pair (S,P), the coalition together with the embedding partition. The set of embedded coalitions is

$$\mathcal{E} = \{(C,\mathcal{P})|C \in \mathcal{P}, \mathcal{P} \in \tilde{\Pi}(N)\}$$

For a set S, an element $i\in S$, and a partition $P=\{S_1,S_2,\dots,S_k\}$ of S, let $P(i)\in P$ such that $i\in P(i)$.

Also, let $P^i(T)=P\backslash\{P(i),T\}\cup\{P(i)^{-i},T^{+i}\}$ denote the partition that results when player i leaves its current coalition and joins coalition T. Let $\Pi^i(P)=\{P^i(T)|T\in P,T\neq P(i)\}$ denote the set of partitions that may result when player i leaves his current coalition and either joins another coalition or stays single.

Let P and Q be two partitions of the same set. The partition R is a common refinement of P and Q if R is a refinement of both P and Q. The coarsest common refinement is denoted by $P\wedge Q$. The operation $\wedge$ as follows (Myerson (1977)):

$$\mathcal{P} \wedge \mathcal{Q} = \{S \cap T|S \in \mathcal{P}, T \in \mathcal{Q}, S \cap T \neq \emptyset\}$$

$P\wedge Q$ is an intermediate partition that captures the intersection of the two original partitions P and Q.

A permutation is a bijection of a set onto itself. For a set N, the set of permutations is $S_n=\{\pi:N\to N|\pi \text{ is bijective}\}$, where a generic element is denoted by π and it can be seen as an ordering of the set N. The permuted image of a particular $i\in S$ element of S is πi. It is natural to consider permutations

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of sets, partitions, or even functions: given a permutation $\pi \in S_n$, the set $S \subseteq N$ is $\pi S=\{\pi i | i \in S\}$. The permutation of a partition $P \in \tilde{\Pi}(S)$ follows naturally: $\pi P=\{\pi S_i \,|S_i \in P\}$. The permutation of an embedded coalition is $\pi(S_i,P)=(\pi S_i,\pi P)$. For $\pi \in S\_n$ and $x=(x_1,\dots,x_n) \in R^n$, $\pi x=(x_{\pi(1)},\dots,x_{\pi(n)})$.

2.1. Integer partitions

A partition nonnegative integer is way of expressing it as the unordered sum of the other positive integers, and it is often written in tuple notation. Formally, $\lambda=[\lambda\_1,\lambda\_2,\dots,\lambda\_l]$ is a partition of n iff $\lambda_1,\lambda_2,\dots,\lambda_l$ are positive integers and $\lambda_1+\lambda_2+\dots+\lambda_l=n$. Two partitions which only differ in the order of their elements are considered to be the same partition. The set of all partitions of n will be denoted by $\Lambda(n)$, and, $\lambda \in \Lambda(n)$. $|\lambda|$ is the number of elements of λ.

For example, the partitions of $n=4$ are $[1,1,1,1]$, $[2,1,1]$, $[2,2]$, $[3,1]$ and $[4]$. Sometimes we will abbreviate this notation by dropping the commas, so $[2,1,1]$ becomes $[211]$.

If $P \in \tilde{\Pi}(N)$, there is a unique partition $\lambda_P \in \Lambda(n)$, associated with P, where the elements of λ_P are exactly the cardinalities of the elements of P. In other words, if $P = \{S_1,S_2,\dots,S_m\} \in \tilde{\Pi}(N)$, then $\lambda\_P=[|S_1|,|S_2|,\dots,|S_m|]$.

For a given $\lambda \in \Lambda(n)$, we represented by λ° the set of the numbers determined by the λ_i's and for $\delta \in \lambda^\circ$, we denoted by m_δ^λ the multiplicity of δ in partition λ with the condition of $m_0^\lambda=1$. So, if $\lambda=[4,2,2,1,1,1]$, then $\lambda^\circ=\{1,2,4\}$ and $m_1^\lambda=3$, $m_2^\lambda=2$, $m_4^\lambda=1$.

Let $\lambda,\gamma \in \Lambda(n)$ be partitions such that $\gamma^\circ \subseteq \lambda^\circ$, we define the difference λ-γ as a new partition obtained from λ by removing the elements of γ. For example, $[4,3,2,1,1,1]$-$[3,1,1]=[4,2,1]$.

2.2. Games in partition function form and the Myerson value

In games with coalition structures, coalitions are part of the usual practice, and multiple coalitions may (and usually do) coexist. For such games, we must define the coalitional values together.

For a partition P, payoffs to coalitions $C \notin P$, including the empty set, are conventionally assumed to be undefined, making the definition elegant but somewhat awkward. More contemporary definitions use embedded coalitions:

Definition 1 *A game in partition function form is a function*

$$w: \mathcal{E} \to \mathbb{R}$$

that assigns a real value to each embedded coalition (S,P), *and such that* $w(\emptyset,P)=0\ \forall P$.

Note that, E representing the set of *embedded coalitions*, is the set of coalitions together with specifications as to how the other players are aligned.

The set of games in partition function form with player set N is denoted by $\tilde{\Gamma}$, i.e.,

$$\tilde{\Gamma} = \{w: \mathcal{E} \to \mathbb{R} | w(\emptyset,\mathcal{P}) = 0 \quad \forall \mathcal{P} \in \tilde{\Pi}(N)\}$$

The value $w(S,P)$ represents the worth of coalition S, given the coalitional structure P. We can see that in games in partition function form (and that in this work, we will only call them *games*), the worth of some coalition S depends not only on what the players of such coalition can obtain together, but also on the way the other players are organized in $N \setminus S$. We assume that, in any game situation, the universal coalition N (embedded in $\{N\}$) will actually form, so that the players will have $w(N,\{N\})$ to divide among themselves.

Now, given w_1, $w_2 \in \tilde{\Gamma}$ and $\alpha \in R$, we can define the sum w_1+w_2 and the product αw_1 in $\tilde{\Gamma}$, in the usual form, i.e,

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$$(w_1 + w_2)(S,\mathcal{P}) = w_1(S,\mathcal{P}) + w_2(S,\mathcal{P}) \quad and \quad (\alpha w_1)(S,\mathcal{P}) = \alpha w_1(S,\mathcal{P})$$

respectively. It is easy to verify that $\tilde{\Gamma}$ is a vector space with these operations.

A *solution* on $\tilde{\Gamma}$ is a function $\varphi:\tilde{\Gamma}\to R^n$. If φ is a solution and $w\in\tilde{\Gamma}$, then we can interpret $\varphi_i(w)$ as the utility payoff which i should expect from de game w.

Myerson's article in 1977 represented a significant achievement by proposing a method for allocating fair contributions in games in partition function form. This method is based on three fundamental axioms, analogous to the Shapley value (Shapley, 1953) for games in characteristic function form, but applied to games expressed in the form of a partition function: linearity, symmetry, and the carrier axiom.

Axiom 1 (Linearity) *The solution* φ *is linear if*

$$\varphi(w_1 + \alpha w_2) = \varphi(w_1) + \alpha\varphi(w_2) \quad \forall w_1, w_2 \in \tilde{\Gamma} \quad \text{and} \quad \alpha \in \mathbb{R}$$

Axiom 2 (Symmetry) *The solution* φ *is said to be symmetric if and only if* $\varphi(\pi w)=\pi\varphi(w)\ \forall\pi\in S_n$, *where the game* πw *is defined as*

$$(\pi w)(S,\mathcal{P}) = w[\pi^{-1}(S,\mathcal{P})], \quad \forall (S,\mathcal{P}) \in \mathcal{E}$$

The linearity axiom tells us how a solution behaves when we linearly combine games, allowing us to calculate the payoff in the combined game as a weighted mixture of solutions in the original games.

Symmetry means that player's payoffs do not depend on their names. The payoff of a player is only derived from his influence on the worth of the coalitions.

Definition 2 *Given* $w\in\tilde{\Gamma}$, *the set* T is *a carrier of* w *if for any embedded coalition* (S,P),

$$w(S,\mathcal{P}) = w(S \cap T, \mathcal{P} \wedge \{T, \overline{T}\})$$

In other words, T is considered a carrier of w if the outcomes obtained by players in S when cooperating in coalition P are the same as what they would achieve when restricted to T. Notice that if a player i is not in T, then $w(\{i\},P)=w(\{i\}\cap T,P)=w(\emptyset,P)=0$, i.e., such player has no influence on the outcome.

The following axiom suggests that the total amount obtained by the grand coalition should be distributed among the members of a carrier:

Axiom 3 (Carrier) *For any* $w\in\tilde{\Gamma}$, *if* T is a carrier of w, *then the following holds*

$$\sum_{i\in T} \varphi_i(w) = w(N,\{N\})$$

Myerson proceeds axiomatically similarly to Shapley and proposes a value that extends the Shapley value. His proposal satisfies the axioms of linearity, symmetry, and carrier.

Theorem 1 (Myerson, 1977) *The solution* $My:\tilde{\Gamma}\to R^n$ *defined as follows*:

$$My_i(w) = \sum_{(S,\mathcal{P})\in\mathcal{E}} (-1)^{|\mathcal{P}|-1}(|\mathcal{P}|-1)! \left[\frac{1}{n} - \sum_{\substack{T\in\mathcal{P}\setminus\{S\} \\ i\notin T}} \frac{1}{(|\mathcal{P}|-1)(n-|T|)}\right] \cdot w(S,\mathcal{P})$$

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where $i \in N$ and $w \in \tilde{\Gamma}$, represents the unique solution that satisfies the axioms of linearity, symmetry, and carrier.

Example 1 *Now, if* $N=\{1,2,3\}$, *according to the Myerson value, the payoff for player 1 is*

$$\begin{aligned} My_1(w) = \; & \tfrac{1}{3} \cdot w(\{1,2,3\},\{\{1,2,3\}\}) \\ & -\tfrac{1}{6} \cdot (2w(\{1\},\{\{1\},\{2\},\{3\}\}) - w(\{2\},\{\{1\},\{2\},\{3\}\}) - w(\{3\},\{\{1\},\{2\},\{3\}\})) \\ & +\tfrac{1}{3} \cdot (2w(\{1\},\{\{1\},\{2,3\}\}) - w(\{2\},\{\{2\},\{1,3\}\}) - w(\{3\},\{\{3\},\{1,2\}\})) \\ & +\tfrac{1}{6} \cdot (w(\{1,2\},\{\{3\},\{1,2\}\}) + w(\{1,3\},\{\{2\},\{1,3\}\}) - 2w(\{2,3\},\{\{1\},\{2,3\}\})) \end{aligned}$$

The Myerson value can be seen as an extension of the well-known Shapley value for games in characteristic function form. Formally, a game in characteristic function form is a function $v:2^N \to R$ such that $v(\emptyset)=0$. Notice that $v(S)$ represents the worth of each coalition S, independently of the partition structure. For this kind of games, the Shapley value Sh is defined as:

$$Sh_i(v) = \sum_{\{S \subseteq N : i \notin S\}} \frac{s!\,(n-s-1)!}{n!} [v(S \cup \{i\}) - v(S)]$$

for each player $i \in N$ and each characteristic function v.

3. NULL PLAYERS IN GAMES IN PARTITION FUNCTION FORM: A NOVEL PROPOSAL

In this section we will introduce a new version of the null player concept, derived from the carrier concept of Myerson (1977). In this way, we will be able to investigate if it is possible to uniquely characterize the Myerson value based on a nullity axiom and an efficiency axiom, similar to the approach used for the Shapley value for games in characteristic function form.

Here, a player $i \in N$ is considered a null player in $w \in \tilde{\Gamma}$ if only if N^{-i} is a carrier (Definition 2) of w.

According to the definition above, if we have a partition P of N, we can observe the following:

1. If $i \in S$, then:

$$\begin{aligned} w(S,\mathcal{P}) & = w(S \cap N^{-i}, \mathcal{P} \wedge \{N^{-i}, N \backslash N^{-i}\}) \\ & = w(S^{-i}, \mathcal{P} \wedge \{N^{-i}, \{i\}\}) \\ & = w(S^{-i}, \{S^{-i}, \{i\}\} \cup \mathcal{P} \backslash \{S\}) \end{aligned}$$

2. If $i \notin S$, then:

$$\begin{aligned} w(S,\mathcal{P}) & = w(S \cap N^{-i}, \mathcal{P} \wedge \{N^{-i}, N \backslash N^{-i}\}) \\ & = w(S, \mathcal{P} \wedge \{N^{-i}, \{i\}\}) \\ & = w(S, \{(\mathcal{P}(i))^{-i}, \{i\}\} \cup \mathcal{P} \backslash \{\mathcal{P}(i)\}) \end{aligned}$$

If $i \notin S$, then $S=S^{-i}$ and if $i \in S$, then $S=P(i)$. This leads us to the following definition

Definition 3 *Given a game in partition function form* w, *a player* $i \in N$ *is a null player in* $w \in \tilde{\Gamma}$ if $\forall (S,P) \in E$

$$w(S,\mathcal{P}) = w(S^{-i}, \{(\mathcal{P}(i))^{-i}, \{i\}\} \cup \mathcal{P} \backslash \{\mathcal{P}(i)\})$$

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This proposal evaluates how the wealth generated by the coalition S varies depending on whether player i is part of that coalition or not. To do this, it adjusts the partition P based on whether i is present or not in the involved coalitions. Removing player i from all coalitions reflects the question of how wealth would be distributed if i were not present. For instance, if $n=4$, $P=\{\{1,2\},\{3,4\}\}$, and player 1 is a null player, then $w(\{1,2\},\{\{1,2\},\{3,4\}\})=w(\{2\},\{\{1\},\{2\},\{3,4\}\})$ and $w(\{3,4\},\{\{1,2\},\{3,4\}\})=w(\{3,4\},\{\{1\},\{2\},\{3,4\}\})$.

Now we are ready to state the following axiom.

Axiom 4 (Nullity) *If* $i\in N$ *is a null player in* $w\in\tilde{\Gamma}$*, then* $\varphi_i(w)=0$.

The nullity axiom only makes sure that a player with absolutely no influence on the gains that any coalition can obtain, should not receive nor pay anything.

In addition and derived from the carrier axiom, we will consider a property where the sum of the individual values of the players in the grand coalition, acting as a carrier, is equal to the total value of the grand coalition. This leads us to establish the following axiom.

Axiom 5 (Efficiency) *For any partition function form game* w,

$$\sum_{i\in N} \varphi_i(w) = w(N,\{N\})$$

This reflects the implicit assumption that players are willing to join the grand coalition, even if some of them are null players who do not receive allocations.

Therefore, the Myerson value satisfies the axioms of linearity, symmetry, nullity and efficiency. However, there are other solutions satisfying the same set of axioms, as shown in the following example.

Example 2 *For* $n=3$*, the solution given by:*

$$\begin{aligned}
\psi_1(w) \; &= \tfrac{1}{3}\cdot w(\{1,2,3\},\{\{1,2,3\}\}) + \sigma\cdot 2w(\{1\},\{\{1\},\{2\},\{3\}\}) \\
&-w(\{2\},\{\{1\},\{2\},\{3\}\}) - w(\{3\},\{\{1\},\{2\},\{3\}\}) - 2w(\{1\},\{\{1\},\{2,3\}\}) \\
&+w(\{2\},\{\{2\},\{1,3\}\}) + w(\{3\},\{\{3\},\{1,2\}\})] \\
&+\tfrac{1}{6}\cdot 2w(\{1\},\{\{1\},\{2,3\}\}) - 2w(\{2,3\},\{\{1\},\{2,3\}\}) + w(\{1,2\},\{\{3\},\{1,2\}\}) \\
&-w(\{3\},\{\{3\},\{1,2\}\}) + w(\{1,3\},\{\{2\},\{1,3\}\}) - w(\{2\},\{\{2\},\{1,3\}\})]
\end{aligned}$$

$$\begin{aligned}
\psi_2(w) \; &= \tfrac{1}{3}\cdot w(\{1,2,3\},\{\{1,2,3\}\}) + \sigma\cdot 2w(\{2\},\{\{1\},\{2\},\{3\}\}) \\
&-w(\{1\},\{\{1\},\{2\},\{3\}\}) - w(\{3\},\{\{1\},\{2\},\{3\}\}) - 2w(\{2\},\{\{2\},\{1,3\}\}) \\
&+w(\{1\},\{\{1\},\{2,3\}\}) + w(\{3\},\{\{3\},\{1,2\}\})] \\
&+\tfrac{1}{6}\cdot 2w(\{2\},\{\{2\},\{1,3\}\}) - 2w(\{1,3\},\{\{2\},\{1,3\}\}) + w(\{1,2\},\{\{3\},\{1,2\}\}) \\
&-w(\{3\},\{\{3\},\{1,2\}\}) + w(\{2,3\},\{\{1\},\{2,3\}\}) - w(\{1\},\{\{1\},\{2,3\}\})]
\end{aligned}$$

$$\begin{aligned}
\psi_3(w) \; &= \tfrac{1}{3}\cdot w(\{1,2,3\},\{\{1,2,3\}\}) + \sigma\cdot 2w(\{3\},\{\{1\},\{2\},\{3\}\}) \\
&-w(\{2\},\{\{1\},\{2\},\{3\}\}) - w(\{1\},\{\{1\},\{2\},\{3\}\}) - 2w(\{3\},\{\{3\},\{1,2\}\}) \\
&+w(\{2\},\{\{2\},\{1,3\}\}) + w(\{1\},\{\{1\},\{2,3\}\})] \\
&+\tfrac{1}{6}\cdot 2w(\{3\},\{\{3\},\{1,2\}\}) - 2w(\{1,2\},\{\{3\},\{1,2\}\}) + w(\{2,3\},\{\{1\},\{2,3\}\}) \\
&-w(\{1\},\{\{1\},\{2,3\}\}) + w(\{1,3\},\{\{2\},\{1,3\}\}) - w(\{2\},\{\{2\},\{1,3\}\})]
\end{aligned}$$

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satisfies the linearity, symmetry, nullity and efficiency axioms, for any choice of the real number σ.

It can be observed that, since $\sigma \in \mathbb{R}$ *is arbitrary, there is an entire class of solutions that fulfill the four axioms, with the Myerson value being a particular case for* $\sigma = -\frac{1}{6}$.

Example 3 *If we assume that there are no externalities, i.e.,*

$$w(\{1\},\{\{1\},\{2\},\{3\}\}) = w(\{1\}\{\{1\},\{2,3\}\}) = v(\{1\})$$
$$w(\{2\},\{\{1\},\{2\},\{3\}\}) = w(\{2\},\{\{2\},\{1,3\}\}) = v(\{2\})$$
$$w(\{3\},\{\{1\},\{2\},\{3\}\}) = w(\{3\},\{\{3\},\{1,2\}\}) = v(\{3\})$$

we have

$$\psi_1(w) = \frac{v(\{1,2,3\})}{3} + \frac{1}{6}[2v(\{1\}) - v(\{2,3\}) + v(\{1,2\}) - v(\{3\}) + v(\{1,3\}) - v(\{2\})]$$

which is the Shapley value for player 1.

For instance, in the following game externalities are not considered:

Table 1. A game with no externalities.

Embedded coalitions	Worth
{1},{2},{3}	(1,4,2)
{1,2},{3}	(6,2)
{1,3},{2}	(4,4)
{2,3},{1}	(8,1)
{1,2,3}	(8)

Thus, $\psi(w) = Sh(v) = \left(1, \frac{9}{2}, \frac{5}{2}\right)$.

4. CHARACTERIZATION OF NULL SOLUTIONS

In this section, we delve into the detailed characterization of the family of solutions that satisfies the axioms of linearity, symmetry, nullity and efficiency. Before delving into the exploration and analysis of these solutions, we will begin by establishing and defining fundamental concepts that will be essential for their understanding.

Let B_n be a set of triples:

$$B_n = \{(\lambda, |S|, |T|) | \lambda \in \Lambda(n) \backslash \{[n]\}, |S| \in \lambda^\circ, |T| \in (\lambda - [|S|])^\circ\}$$

Let F_n be a set of pairs:

$$F_n = \{(\lambda, |S|) | \lambda \in \Lambda(n), |S| \in \lambda^\circ \backslash \{1, n\}\}$$

For $\lambda \in \Lambda(n)$, $\delta \neq 1$ and $\delta, \epsilon \in \lambda$, we define,

$$\lambda_\epsilon^\delta = \lambda - [\delta, \epsilon] + [\epsilon + 1, \delta - 1]$$

and

$$\lambda_0^\delta = \lambda - [\delta] + [1, \delta - 1]$$

Example 4 *If* $n=4$*, then*

$$B_4 = \{([1111],1,1),([211],1,1),([211],1,2),([211],2,1),([22],2,2),([31],1,3),([31],3,1)\}$$

If $n=5$*, then*

$$F_5 = \{([2111],2),([221],2),([311],3),([32],2),([32],3),([41],4)\}$$

For instance, if $(\lambda,|S|)=([2111],2)$*, we have* $m_2^\lambda = 1,\ \lambda_0^2 = [11111]$ *and* $\lambda_1^2 = [2111]$.

Additionally, let $\beta=\{\beta_{(\lambda,|S|,|T|)}\ |(\lambda,|S|,|T|)\in B_n\}$ be a set of parameters indexed by B_n and we define for any embedded coalition (S,P) such that $S\neq N$ and for any fixed $w\in\tilde{\Gamma}$, the quantity:

$$A_i^\beta(S,\mathcal{P}) = \begin{cases} \sum\limits_{T\in\mathcal{P}\setminus\{S\}} |T|\beta_{(\lambda_\mathcal{P},|S|,|T|)} \cdot w(S,\mathcal{P}) & if\ i\in S \\ -|S|\beta_{(\lambda_\mathcal{P},|S|,|\mathcal{P}(i)|)} \cdot w(S,\mathcal{P}) & if\ i\notin S \end{cases}$$

Next, we provide the characterization result.

Theorem 2 *The solution* $\psi:\tilde{\Gamma}\to R^n$ *satisfies linearity, symmetry, nullity and efficiency axioms if only if it is of the form*

(1)

$$\psi_i(w) = \frac{w(N,\{N\})}{n} + \sum_{\substack{(S,\mathcal{P})\in\mathcal{E} \\ S\neq N}} A_i^\beta(S,\mathcal{P})$$

for some real numbers $\beta=\{\beta_{(\lambda,|S|,|T|)}\ |(\lambda,|S|,|T|)\in B_n\}$ *such that*

(2)

$$\beta_{([n-1,1],n-1,1)} = \frac{1}{n(n-1)}$$

and

For every $(\lambda,\delta)\in F_n$:

(3)

$$\sum_{\substack{\epsilon\in\lambda^\circ\cup\{0\} \\ \epsilon\neq\delta}} \left[\delta m_\epsilon^\lambda \beta_{(\lambda,\delta,\epsilon)} - (\delta-1) m_\epsilon^\lambda \beta_{(\lambda_\epsilon^\delta,\delta-1,\epsilon+1)}\right] = \left(m_\delta^\lambda - 1\right)\left[(\delta-1)\beta_{(\lambda_\delta^\delta,\delta-1,\delta+1)} - \delta\beta_{(\lambda,\delta,\delta)}\right]$$

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Moreover, such representation is unique.

Proof. First, let ψ be a solution on $\tilde{\Gamma}$ that satisfies the four axioms. From Sánchez-Pérez (2023, Theorem 1), if ψ is linear, symmetric and efficient, then there exists a unique sequence $\beta=\{\beta_{(\lambda,|S|,|T|)} \,|(\lambda,|S|,|T|)\in B_n\}$ such that for any $i\in N$,

$$\psi_i(w) = \frac{w(N,\{N\})}{n} + \sum_{\substack{(S,\mathcal{P})\in\mathcal{E}\\ S\neq N}} A_i^{\beta}(S,\mathcal{P})$$

Now, consider a basis $\{u_{(S,P)} \,|(S,P)\in E, S\neq\emptyset\}$ for $\tilde{\Gamma}$, where the games $u_{(S,P)}$ are defined by

$$u_{(S,\mathcal{P})}(T,\mathcal{Q}) = \begin{cases} 1 & \text{if } S\subseteq T \text{ and } \mathcal{P}\wedge\mathcal{Q}=\mathcal{P} \\ 0 & \text{otherwise} \end{cases}$$

Notice that $i\in N$ is a null player in the game $u_{\left(N^{-i},\left\{N^{-i},\{i\}\right\}\right)}$, since

$$u_{\left(N^{-i},\left\{N^{-i},\{i\}\right\}\right)}(S,\mathcal{P}) - u_{\left(N^{-i},\left\{N^{-i},\{i\}\right\}\right)}\left(S^{-i},\left\{S^{-i},\{i\}\right\}\cup\mathcal{P}\backslash\{S\}\right) = 0$$

and

$$u_{\left(N^{-i},\left\{N^{-i},\{i\}\right\}\right)}(S,\mathcal{P}) - u_{\left(N^{-i},\left\{N^{-i},\{i\}\right\}\right)}(S,\{(\mathcal{P}(i))^{-i},\{i\}\}\cup\mathcal{P}\backslash\{\mathcal{P}(i)\}) = 0$$

for every $(S,P)\in E$.

The nullity axiom implies:

$$0 = \psi_i(u_{\left(N^{-i},\left\{N^{-i},\{i\}\right\}\right)}) = \frac{1}{n} - (n-1)\beta_{([n-1,1],n-1,1)}$$

Hence,

$$\beta_{([n-1,1],n-1,1)} = \frac{1}{n(n-1)}$$

Once again, the nullity axiom implies

(4)

$$0 = \psi_i(u_{(S,\mathcal{P})}) = \sum_{T\in\mathcal{P}\backslash S}\left[|T|\beta_{(\lambda_{\mathcal{P}},|S|,|T|)} - (|S|-1)\beta_{(\{\mathcal{P}^i(T),T^{+i}\}\cup\mathcal{P}\backslash\{S,T\},|S|-1,|T|+1)}\right]$$

for every pair $(S,P)\in E$ such that $|S|\notin\{1,n\}$ and $i\in S$.

Notice that relation (4) yields many repeated equations. In particular, such relation provides the same equation for (S,P) and (S',P'), if $|S|=|S|'$ and $\lambda_P=\lambda_{P'}$. Therefore, the number of distinct equations derived from (4) coincides with the number of elements in F_n.

On the other hand, for a fixed (S,P) such that $|S|\notin\{1,n\}$, it holds:

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(5)

$$\sum_{T\in\mathcal{P}\backslash\{S\}} |T|\beta_{(\lambda_{\mathcal{P}},|S|,|T|)} = \begin{cases} \left(m_{|S|}^{\lambda_{\mathcal{P}}} - 1\right)|S|\beta_{(\lambda_{\mathcal{P}},|S|,|S|)} & if\ |T| = |S| \\ \sum\limits_{\substack{\epsilon\in\mathring{\lambda}_{\mathcal{P}}\cup\{0\}\\ \epsilon\neq|S|}} \epsilon m_{\epsilon}^{\lambda_{\mathcal{P}}}\beta_{(\lambda_{\mathcal{P}},|S|,\epsilon)} & if\ |T| \neq |S| \end{cases}$$

and also

(6)

$$\sum_{T\in\mathcal{P}\{S\}} \beta_{(\{\mathcal{P}^i(T),T^{+i}\}\cup\mathcal{P}\{S,T\},|S|-1,|T|+1)} = \begin{cases} \left(m_{|S|}^{\lambda_{\mathcal{P}}} - 1\right)\beta_{\left((\lambda_{\mathcal{P}})_{|S|}^{|S|},|S|-1,|S|+1\right)} & if|T| = |S| \\ \sum\limits_{\substack{\epsilon\in\mathring{\lambda}_{\mathcal{P}}\cup\{0\}\\ \epsilon\neq|S|}} m_{\epsilon}^{\lambda_{\mathcal{P}}}\beta_{\left((\lambda_{\mathcal{P}})_{\epsilon}^{|S|},|S|-1,\epsilon+1\right)} & if|T| \neq |S| \end{cases}$$

The substitution of equalities (5) and (6) in relation (4) yields relation (3). The converse is a straightforward computation.

In order to check uniqueness it is enough to prove that if

$$0 = \frac{w(N,\{N\})}{n} + \sum_{\substack{(S,\mathcal{P})\in\mathcal{E}\\ S\neq N}} A_i^{\beta}(S,\mathcal{P})$$

$$= \frac{w(N,\{N\})}{n} + \sum_{(\lambda,|S|,|T|)} \beta_{(\lambda,|S|,|T|)}\left[\sum_{\substack{(S,\mathcal{P})\in\mathcal{E}\\ \lambda_{\mathcal{P}}=\lambda,S\ni i}} \sum_{T\in\mathcal{P}\backslash\{S\}} |T|w(S,\mathcal{P}) - \sum_{\substack{(S,\mathcal{P})\in\mathcal{E}\\ \lambda_{\mathcal{P}}=\lambda,S\not\ni i\\ |\mathcal{P}(i)|=|T|}} |S|w(S,\mathcal{P})\right]$$

where the $\beta_{(\lambda,|S|,|T|)}$'s satisfy conditions (2) and (3) for $(\lambda,\delta)\in F_n$, for every game w and for every player i, then every $\beta_{(\lambda,|S|,|T|)}$ vanish.

Thus, for given $(\lambda,|S|,|T|)\in B_n$ let $S=\{1,\ldots,s\}$ and P be any partition such that $S\in P$ and $\lambda_P=\lambda$. Also let $T\in P$. Let $w=u_{(S,P)}$ and pick any $i\in T$. Then the above sum reduces to

$$0 = \beta_{(\lambda,|S|,|T|)}$$

Example 5 *For* $n=4$*, every solution satisfying the axioms of linearity, symmetry, nullity and efficiency takes the form given in (1), where the parameters* β*'s satisfy* $\beta_{([31],3,1)} = \frac{1}{12}$ *and from the system of equations (3):*

$$-\beta_{([1111],1,1)} + 2\beta_{([211],2,1)} - 2\beta_{([211],1,2)} = 0$$
$$2\beta_{([22],2,2)} - \beta_{([31],1,3)} - \beta_{([211],1,1)} = 0$$
$$-2\beta_{([211],2,1)} + \beta_{([31],3,1)} - 2\beta_{([22],2,2)} = 0$$

Of course, the Myerson value meet the constraints established by the above system of equations. Indeed, its corresponding parameters are the following:

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Table 2. Parameters for the Myerson value.

β([1111],1,1)	β([211],1,2)	β([211],1,1)	β([211],2,1)	β([22],2,2)	β([31],1,3)	β([31],3,1)
1/6	-1/6	0	-1/12	1/8	1/4	1/12

Corollary 1 The space of all linear, symmetric, efficient and null solutions in n players has dimension $|B_n|-|F_n|-1$.

Now, a natural requirement for a fair division scheme is that it remunerates the players of a game taking into account their contribution to the surplus generated via cooperation. Indeed, in Shapley's axiomatization for games in characteristic function form (externalities are not considered), the nullity axiom requires that no share be allocated to players with zero contribution to any possible coalition that could be created in the coalitional game. The key issue, then, is how such contribution should be measured.

Here for environments with externalities, we propose to define the marginal contribution of a player i in the context of the partition function $w \in \tilde{\Gamma}$ as follows: Given a partition function $w \in \tilde{\Gamma}$ and a coalition $S \subseteq N$, the marginal contribution of player i in the coalition S is defined as:

$$MC_i(S,\mathcal{P}) = w(S,\mathcal{P}) - w(S^{-i},\{(\mathcal{P}(i))^{-i},\{i\}\} \cup \mathcal{P}\backslash\{\mathcal{P}(i)\})$$

The marginal contribution of a player $i \in N$ to $(S,P) \in E$ can be understood in two aspects:

1. Direct ($i \in S$, i.e., $S = \mathcal{P}(i)$):

$$MC_i(S,\mathcal{P}) = w(S,\mathcal{P}) - w(S^{-i},\{S^{-i},\{i\}\} \cup \mathcal{P}\backslash\{S\})$$

2. Indirect ($i \notin S$):

$$MC_i(S,\mathcal{P}) = w(S,\mathcal{P}) - w(S,\{(\mathcal{P}(i))^{-i},\{i\}\} \cup \mathcal{P}\backslash\{\mathcal{P}(i)\})$$

The direct marginal contribution evaluates how the presence of player i in the coalition S affects the distribution of wealth in that specific coalition. It compares the original distribution of wealth in S with the adjusted distribution that would result if i decided to leave the coalition and work alone. If the difference is positive, the presence of i improves the distribution of wealth in S; if negative, it could be negatively affecting that distribution. In summary, MC_i quantifies the direct impact of player i participating in the wealth of coalition S.

In the case of indirect marginal contribution, if i is not in S, we want to evaluate how the situation would be affected if i decided to join. To do this, we consider how the total benefit of the team in S changes when i leaves another team, $T \in P$ (which is not S), and decides to work alone. This indirect marginal contribution helps us understand the potential impact of i's participation in S by considering their exit from another team.

In essence, we are exploring how the presence or absence of i can influence the group's benefits, either directly when they are already part of the team or indirectly when evaluating how their decision to join would affect other teams. The structure of the partition is adjusted to isolate i and evaluate this impact more clearly.

So, the marginal contribution represents the total wealth generated by the coalition S that is distributed among the players according to the partition function w. The marginal contribution of i in S is calculated by taking the difference between the original distribution and the adjusted distribution when i joins or leaves the coalition.

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This quantity reflects the specific influence of player i on the wealth distribution in coalition S. If i has no influence (because he is a null player), the marginal contribution will always be zero, indicating that their presence or absence does not affect the wealth distribution in that coalition.

In this context, it is important to highlight that it could be possible to express any linear, symmetric, null and efficient solution in partition function games as a linear combination of marginal contributions. By focusing on the payment of players in this way, it provides a clearer and more accessible insight into how the individual actions of each player can marginally affect the final outcome of the game.

Example 6 *From example 2, the expression* $\psi_1(w)$ *represents the payment assigned to player 1, computed from a partition function* w *in the case* $n=3$. *Each term in the formula reflects the contribution of player 1 in different scenarios, considering the coalitions in which they participate. The classification as direct or indirect indicates whether the contribution is direct when present in the coalition or indirect if not.*

$$
\begin{array}{lr}
\psi_1(w) = 2\sigma[w(\{1\},\{\{1\},\{2\},\{3\}\}) - w(\emptyset,\mathcal{P})] & Direct \\
+(\frac{1}{3}-\sigma)[w(\{1\},\{\{1\},\{2,3\}\}) - w(\emptyset,\mathcal{P})] & Direct \\
+\frac{1}{6}[w(\{1,2\},\{\{1,2\},\{3\}\}) - w(\{2\},\{\{1\},\{2\},\{3\}\}] & Direct \\
+\frac{1}{6}[w(\{1,3\},\{\{1,3\},\{2\}\}) - w(\{3\},\{\{1\},\{2\},\{3\}\}] & Direct \\
+(\sigma-\frac{1}{6})[w(\{2\},\{\{1,3\},\{2\}\}) - w(\{2\},\{\{1\},\{2\},\{3\}\}] & Indirect \\
+(\sigma-\frac{1}{6})[w(\{3\},\{\{1,2\},\{3\}\}) - w(\{3\},\{\{1\},\{2\},\{3\}\}] & Indirect \\
+\frac{1}{3}[w(\{1,2,3\},\{\{1,2,3\}\}) - w(\{2,3\},\{\{1\},\{2,3\}\}] & Direct
\end{array}
$$

Recall that this solution satisfies the axioms of linearity, symmetry, nullity and efficiency for any choice of $\sigma \in R$.

Remark 1 *It is well known that in the Shapley context, given a carrier coalition* T, *if* $i \notin T$, *then* i *is considered a null player. However, this is not necessarily the case with the Myerson carrier in* $\tilde{\Gamma}$, *as players may have an individual value but do not generate surplus when joining coalitions.*

Notice that our definition of a null player and Myerson's carrier definition reveal that if a player is not in a carrier T, it does not necessarily mean he is null. This will be better visualized in the following example.

Example 7 *Consider the following game*:

Table 3. A game with a carrier.

Embedded coalitions	Worth
{1},{2},{3}	(5,0,0)
{1,2},{3}	(5,0)
{1,3},{2}	(5,0)
{2,3},{1}	(0,7)
{1,2,3}	(7)

According to Definition 2, coalition $T=\{1\}$ *is a carrier for the previous game. However, players 2 and 3 are not null players (Definition 3).*

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The payoff for players, from the solution $\psi(w)$ that satisfies the axioms of linearity, symmetry, nullity and efficiency, is

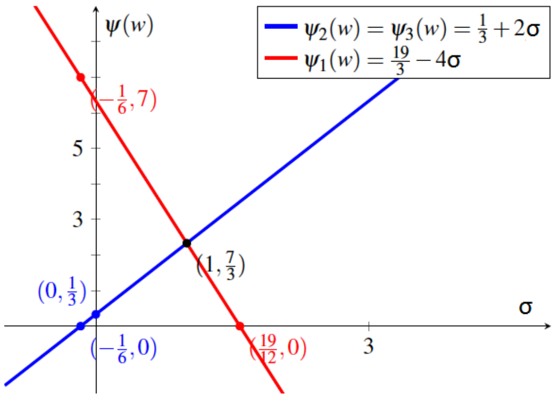
(7)

$$\psi(w) = \left(\frac{19}{3} - 4\sigma, \frac{1}{3} + 2\sigma, \frac{1}{3} + 2\sigma\right)$$

where $\sigma \in R$ is arbitrary.

The following graph shows this family of solutions:

Figure 1. Family of linear, symmetric, efficient and null solutions for 3 players.



In this context, the payments along the lines represent the payment assignments for the players as a function of the parameter σ. Each line in the graph corresponds to the payments of a specific player, and its slope determines how those payments change as we adjust the value of σ.

1) Blue lines (players 2 and 3): Both blue lines are the same, indicating that players 2 and 3 receive the same payment amount. The slope of these lines is positive, indicating that the payments of players 2 and 3 increase as σ increases. When $\sigma=0$, both players receive a payment of $\frac{19}{3}$, and this payment increases linearly with σ.

2) Red line (player 1): The red line has a negative slope, meaning that the payment of player 1 decreases as σ increases. When $\sigma=0$, player 1 receives a payment of $\frac{19}{3}$, and this payment decreases linearly with σ.

In general, moving along these lines in the graph reflects how the payment assignments for each player evolve consistently in response to changes in the parameter σ according to the solution given by equation (7), causing the multiplicity of solutions.

4.1. Comparison with other null players definitions

Now we turn to different versions of players who do not contribute to the game, commonly referred in related literature to as null players or *dummy players*. There are various iterations of null players, and each version gives rise to a different extension of the Shapley value.

Definition 4 (Null player in partition function form games (Pham Do and Norde, 2007)) Given a partition function form game (N,w), a player i is a null player if for all $P \ni \{i\}$,

- $w(\{i\},P)=0$ and
- $w(T^{+i},P^{i}(T))=w(T,P) \quad \forall T \in P$

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Our definition:

- *It focuses on the redistribution of wealth in the game when the null player is present or absent in a coalition.*
- *It evaluates how the partition* $\mathcal{P}$ *changes when the null player is involved in a coalition and when they are not.*
- *It does not establish specific conditions on the value of the null player or their impact on other coalitions.*

Definition from Pham Do and Norde (2007):

- It is based on two distinct conditions:

a) The null player has no value by himself and does not contribute to any coalition they participate in.

b) The wealth generated in a coalition before and after the null player leaves their coalition and joins another is equal.

- Does not consider the impact of the null player on the redistribution of wealth between coalitions but rather focuses on their nullity in terms of contribution and maintaining constant wealth in the coalitions, without considering their impact on redistribution.

In summary, the main difference lies in the focus of each definition: our proposal focuses on how the null player affects the distribution of wealth, while the definition from the literature focuses on the nullity of the player in terms of contribution and on maintaining constant wealth in the coalitions, without considering their impact on redistribution.

Definition 5 (Dummy player (Macho-Stadler et al., 2007)) *Given a game in partition function form* w*, a player* i *is a null player if for all* $\mathcal{P}\ni\{i\}$*,*

$$w(T^{+i},\mathcal{P}^i(T)) = w(T,\mathcal{P}) \quad \forall T \in \mathcal{P}$$

Note that Pham Do and Norde require the wealth associated with the coalition $\{i\}\in\mathcal{P}$ to be zero for a player to be considered null, while Macho-Stadler does not include this additional condition. Both definitions share the consistency condition of wealth in coalitions that involve the null player.

Definition 6 (Efficient-cover null player (Hafalir, 2007)) *Given a game in partition function form* w*, a player* i *is an efficient-cover null player if for all* $S\subseteq N$*, the efficient-cover function* $\overline{w}$ *satisfies*

$$\overline{w}(S,\{S\}\cup\overline{S}]) = \overline{w}(S^{-i},\{S^{-i}\}\cup\overline{S}]^{+i})$$

Our definition:

- It directly applies to a partition function in a game $(N,\mathcal{P})$.
- The comparison focuses on how wealth is distributed across different coalitions with or without the presence of player i.
- It does not make explicit assumptions about the structure of the remaining players in the coalitions.

Hafalir's definition (Efficient-Cover Null Player):

- It applies to the efficient partition of a game and its partition function $\overline{w}$.
- The comparison is made in terms of how the efficient partition behaves rather than the original partition.
- It explicitly assumes that the remaining players in the coalition are singletons (individual players without coalitions).

In summary, the main difference lies in the focus and the context of application. Our proposal centers on the original partition function and applies more generally to different coalition structures, without making assumptions about the remaining players. In contrast, Hafalir's definition focuses on the efficient partition and assumes a specific structure of the remaining players (all are singletons) in the coalitions.

Definition 7 (Null player in the strong sense (Bolger, 1989)) *Given a game in partition function form* w*, a player* i *is a null player in the strong sense if for all* $(S,P)\in E$ *and* $T\in P$ *where* $T\neq S$,

$$w(S,\mathcal{P}) = w(S^{-i},\mathcal{P}^{i}(T))$$

Therefore, the value of a coalition is not changed if a null player in the strong sense is transferred to another coalition in the partition. As a special case, looking at $S=\{i\}$ shows that the value of a singleton that is a null player in the strong sense is zero in any coalition structure and, therefore, the term null is more appropriate. Notice also that $i\in S$ is not required. In essence, this property captures the irrelevance of null players (Macho-Stadler et al., 2007) in two aspects:

1. A null player in the strong sense does not contribute to a coalition since if $i\in S$, then the departure of i does not change the payoff of S. (Using the relation on T^{+i} also shows that i does not increase the payoff of T by joining.)
2. A null player in the strong sense does not generate externalities since moving i between coalitions does not change the payoff of third parties, that is, coalitions $S\neq\{i\}$.

The concept of a null player in the strong sense is intimately related to extended carriers. If $C\subseteq N$ and C^{-i}, where $i\in C$, are both extended carriers, then i is a null player in the strong sense (McQuillin, 2009).

Now,

- **Nullity condition:** In our proposal, a player i is considered a null player if, for every pair (S,P) in the efficient coalition structure, the wealth generated by the coalition S remains the same, regardless of whether i is present or absent in that coalition. In Bolger's definition, a player i is a null player in the strong sense if, for every pair (S,P) and for every set T in P (including the possibility of transferring i to other coalitions), the wealth of S is invariant. This implies that a strong null player not only doesn't affect the coalition to which they belong but also does not affect other coalitions to which they might be transferred.
- **Contribution to coalitions:** In our proposal, it is emphasized that a null player does not contribute to a coalition since their presence or absence does not affect the wealth distribution. In Bolger's definition, this is reflected in the property that the exit of a strong null player (if $i\in S$) does not change the outcome of coalition S, indicating that this player does not contribute to the coalition in terms of wealth.
- **Generation of externalities:** Our proposal does not specifically mention externalities, but Bolger's definition addresses this aspect by stating that a strong null player does not generate externalities since moving i between coalitions does not affect the rewards of third parties (coalitions $S\neq\{i\}$).
- **Presence of player i:** In our proposal, it is not required for i to be present in coalition S to be considered a null player. Bolger's definition does not mention the need for i to be present in S but focuses on how their transfer affects rewards.

Definition 8 (Null player with steady marginality (Skibski, 2011)) *A player* i *is a null player with steady marginality if for all partitions* $P\in\tilde{\Pi}(N)$,

(8)

$$\sum_{\substack{T\in\mathcal{P}\\ T\ni i}} w(\mathcal{P}(i),\mathcal{P}) - w(\mathcal{P}(i)^{-i},\mathcal{P}^{i}(T)) = 0 \quad \text{and}$$

(9)

$$w(N,\{N\}) - w(N^{-i},\{N^{-i},\{i\}\}) = 0$$

The definition of *null player with steady marginality* establishes a special condition for a player i in a cooperative game. It claims that a player that is a null player in the strong sense is also a player with steady marginality, so, a player is said to be a null player with steady marginality if, for all ways of partitioning the players into coalitions, a specific condition holds.

This condition involves two parts:

1. The first part of the condition (8) states that, for each partition P that does not include player i in any coalition T, the sum of differences between two wealth values must be equal to zero. These wealth values are associated with player i participating or not participating in coalition $P(i)$.
2. The second part of the condition states that the wealth difference between the total game N and the game excluding player i must also be equal to zero. This implies that player i's contribution to the total game and their contribution when excluding i along with a coalition containing only i must be equivalent.

In summary, this definition characterizes a null player with steady marginality as one whose contribution to wealth is consistent and balanced across all possible partitions of the player set.

- Difference in formulation: In our proposal, the condition for a player to be null is based on an equality that compares the wealth of a coalition with the wealth of the same coalition after removing player i. In contrast, Skibski's definition is based on two separate conditions: one related to the sum of wealth differences between coalitions in a partition and another related to the wealth of the entire set of players.
- Steady marginality: Skibski's definition incorporates the concept of steady marginality, which implies that player i cannot join an empty coalition and form a new coalition. This restriction is not present in our proposal.
- Relationship between definitions: Skibski points out that a null player in the strong sense (satisfying the conditions of our definition) is also a player with constant marginality. However, the two definitions are not mutually related, indicating a difference in approach and requirements to be a null player.

5. CONCLUSION

In summary, this work has focused on the quest for alternative characterization of Myerson's value in partition function form games. Unlike Shapley, whose uniqueness is established in characteristic function games, we have demonstrated that in partition function form games, it is not possible to find a characterization that is simultaneously linear, symmetric, null and efficient. This finding highlights the fundamental structural differences between these two types of cooperative games.

The introduction of a null player in the characterization reveals a family of parameterized solutions, indicating the diversity of possible outcomes and the lack of uniqueness in this context. However, to achieve a unique solution in partition function form games, it is suggested to explore additional axioms or consider more specific axioms that can constrain the parameterization.

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One possible avenue to attain uniqueness could be the introduction of additional constraints on the alternative parameters, such as specific conditions on their values or relationships between them. These constraints could be derived from desirable additional properties in the specific problem context or from empirical analyses of concrete situations.

In conclusion, this work not only provides a clear insight into the limitations of a unique characterization in partition function form games but also suggests that the pursuit of uniqueness could benefit from the inclusion of additional constraints based on more detailed theoretical considerations or practical application of the model. These suggestions offer a valuable direction for future research, opening new opportunities to better understand the nature of solutions in cooperative games.

AUTHOR CONTRIBUTIONS

Conceptualization, JSP; data curation JSP and ALM; formal analysis, JSP and ALM; methodology, JSP and ALM; software, JSP and ALM; validation, JSP and ALM; investigation, JSP and ALM; resources, JSP and ALM; writing original draft preparation, JSP and ALM; writing review and editing, JSP; visualization, JSP; project administration, JSP; funding acquisition, JSP.

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