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# The Generalized Hybrid Averaging Operator and its Application in Decision Making

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## ABSTRACT

We present the generalized hybrid averaging (GHA) operator. It is a new aggregation operator that generalizes the hybrid averaging (HA) operator by using the generalized mean. Thus, we are able to generalize a wide range of mean operators such as the HA, the hybrid geometric averaging (HGA), the hybrid quadratic averaging (HQA), the generalized ordered weighted averaging (GOWA) operator and the weighted generalized mean (WGM). A key feature in this aggregation operator is that it is able to deal with the weighted average and the ordered weighted averaging (OWA) operator in the same formulation. We further generalize the GHA by using quasi-arithmetic means obtaining the quasi-arithmetic hybrid averaging (Quasi-HA) operator. We conclude the paper with an example of the new approach in a financial decision making problem.

**Keywords:** aggregation operator; OWA operator; generalized mean; weighted average; decision making.
**JEL classification:** C44; C49; D81; D89.
**MSC2010:** 90B50.

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# La media generalizada híbrida y su aplicación en la toma de decisiones


## RESUMEN

En este artículo se presenta el operador de medias generalizadas híbridas. Es un nuevo operador de agregación que generaliza la media híbrida utilizando la media generalizada. Debido a esto, se puede generalizar una amplia gama de operadores de medias, como la media híbrida, la media geométrica híbrida, la media cuadrática híbrida, la media ponderada ordenada generalizada y la media ponderada generalizada. Un aspecto fundamental en este operador de agregación es la posibilidad de utilizar medias ponderadas y medias ponderadas ordenadas en la misma formulación. A continuación, se presenta una generalización mayor mediante la utilización de medias cuasi-aritméticas, obteniendo así la media cuasi-aritmética híbrida. El trabajo termina con un ejemplo de aplicación del nuevo modelo en un problema de toma de decisiones financieras.



**Palabras clave:** operador de agregación; operador OWA; media generalizada; media ponderada; toma de decisiones.
**Clasificación JEL:** C44; C49; D81; D89.
**MSC2010:** 90B50.

## 1. INTRODUCTION

Different types of aggregation operators are found in the literature for aggregating the information. A very common aggregation method is the ordered weighted averaging (OWA) operator (Yager, 1988). It provides a parameterized family of aggregation operators that includes as special cases the maximum, the minimum and the average criteria. Since its appearance, the OWA operator has been used in a wide range of applications (Beliakov *et al*., 2007; Calvo *et al*. 2002; Canós and Liern, 2008; Merigó, 2008; Xu, 2005; Xu and Da, 2002; Yager, 1993; 1996a; 2002; 2008; Yager and Kacprzyk, 1997).

In 2003, Xu and Da introduced the hybrid averaging (HA) operator. It is an aggregation operator that uses the weighted average (WA) and the OWA operator in the same formulation. Then, it is possible to consider in the same problem the attitudinal character of the decision maker and the subjective probablity. For further research on the HA operator, see Merigó, 2008; Wei, 2009; Xu, 2004; 2009; Zhao *et al*., 2010.

Another interesting aggregation operator is the generalized OWA (GOWA) operator (Karayiannis, 2000; Yager, 2004). It generalizes the OWA operator by using generalized means (Dyckhoff and Pedrycz, 1984). Then, it includes as special cases, the maximum, the minimum and the average criteria, and a wide range of other means such as the OWA operator itself, the ordered weighted geometric (OWG) operator (Herrera *et al*., 2003), the ordered weighted quadratic averaging (OWQA) operator, etc. The GOWA operator has been further generalized by using quasi-arithmetic means (Beliakov, 2005) obtaining the Quasi-OWA operator (Fodor *et al*., 1995). For further research on the GOWA operator, see Beliakov *et al*., 2007; Calvo *et al*., 2002; Merigó, 2008; Merigó and Casanovas, 2010, Merigó and Gil-Lafuente, 2008; 2009a; 2009b; Zhao *et al*., 2010.

In this paper, we introduce the generalized hybrid averaging (GHA) operator. It generalizes the HA operator by using generalized means. Then, it includes in the same formulation all the cases coming from the generalized mean such as the arithmetic mean, the geometric mean, the quadratic mean, etc, and a lot of other cases such as the weighted generalized mean (WGM) and the generalized ordered weighted averaging (GOWA) operator. We also obtain new aggregation operators such as the hybrid geometric averaging (HGA) operator, the hybrid quadratic averaging (HQA) operator, the hybrid harmonic averaging (HHA) operator, etc. We further generalize the GHA operator by using quasi-arithmetic means, obtaining the quasi-HA operator.

Moreover, we present an example of the application of the new approach in a financial decision making problem where we can see how it can be implemented in the real life. The main advantage of using the GHA is that it gives a more complete view of the problem to the decision maker because it generalizes a wide range of mean operators allowing the decision maker to select the particular type that it is in closest accordance with his interests..

In order to do so, this paper is organized as follows. In Section 2, we briefly review some basic aggregation operators. In Section 3, we present the GHA operator. Section 4 studies different families of GHA operators. In Section 5, we introduce the Quasi-HA operator. Section 6 develops an application of the new approach in a financial decision making problem. Finally, in Section 7 we summarize the main conclusions found in the paper.

## 2. PRELIMINARIES

In this Section we briefly describe the hybrid averaging (HA) operator and the generalized OWA (GOWA) operator.

### 2.1 The Hybrid Averaging Operator

The HA operator (Xu and Da, 2003) is an aggregation operator that uses the WA and the OWA operator in the same formulation. Then, it is possible to consider in the same problem, the attitudinal character of the decision maker and its subjective probability. One of its main characteristics is that it provides a parameterized family of aggregation operators that includes the maximum, the minimum, the arithmetic mean (AM), the WA and the OWA operator. It can be defined as follows.

**Definition 1.** An HA operator of dimension $n$ is a mapping $HA$: $R^n \rightarrow R$ that has an associated weighting vector $W$ of dimension $n$ with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, such that:

$$HA\ (a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j\ , \tag{1}$$

where $b_j$ is the $j^{\text{th}}$ largest of the $\hat{a}_i$ ($\hat{a}_i = n\omega_i a_i$, $i = 1,2,\ldots,n$), $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weighting vector of the $a_i$, with $\omega_i \in [0, 1]$ and the sum of the weights is 1.

From a generalized perspective of the reordering step, we can distinguish between the descending HA (DHA) operator and the ascending HA (AHA) operator. The weights of these operators are related by $w_j = w^*_{n-j+1}$, where $w_j$ is the $j$th weight of the DHA and $w^*_{n-j+1}$ the $j$th weight of the AHA operator.

Note that different families of HA operators are found by using a different manifestation in the weighting vector such as the step-HA operator, the window-HA operator, the median-HA operator, the centered-HA operator, etc (Merigó, 2008).

### 2.2 The Generalized OWA Operator

The GOWA operator (Karayiannis, 2000; Yager, 2004) is a generalization of the OWA operator by using generalized means. It includes a wide range of means such as the arithmetic mean (AM), the OWG operator, etc. It can be defined as follows.

**Definition 2.** A GOWA operator of dimension $n$ is a mapping $GOWA$: $R^n \rightarrow R$ that has an associated weighting vector $W$ of dimension $n$ with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, such that:

$$GOWA\ (a_1, a_2, \ldots, a_n) = \left( \sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda}, \tag{2}$$

where $b_j$ is the $j^{\text{th}}$ largest of the $a_i$, and $\lambda$ is a parameter such that $\lambda \in (-\infty, \infty)$.

From a generalized perspective of the reordering step, it is possible to distinguish between the descending generalized OWA (DGOWA) operator and the ascending generalized OWA (AGOWA) operator. The weights of these operators are related by $w_j = w^*_{n-j+1}$, where $w_j$ is the $j$th weight of the DGOWA and $w^*_{n-j+1}$ the $j$th weight of the AGOWA operator.

As it is explained by Yager (2004), the GOWA operator is monotonic, commutative, bounded and idempotent. It can also be demonstrated that the GOWA operator has as special cases the maximum, the minimum, the generalized mean and the weighted generalized mean, among others. Other families of GOWA operators are found in Merigó, 2008; Merigó and Gil-Lafuente, 2009b, such as the step-GOWA operator, the olympic-GOWA and the S-GOWA operator.

If we analyze different values of the parameter $\lambda$, we can also obtain other special cases of GOWA operators such as the usual OWA operator, the OWG operator, the ordered weighted harmonic averaging (OWHA) operator and the ordered weighted quadratic averaging (OWQA) operator. When $\lambda = 1$, we obtain the usual OWA operator. When $\lambda = 0$, we get the OWG operator. When $\lambda = -1$, the OWHA operator. When $\lambda = 2$, the OWQA operator.

If we replace $b^{\lambda}$ with a general continuous strictly monotone function $g(b)$, then, the GOWA operator becomes the Quasi-OWA operator (Fodor *et al*., 1995). It can be formulated as follows.

**Definition 3.** A Quasi-OWA operator of dimension n is a mapping *QOWA*: $R^n \rightarrow$ R that has an associated weighting vector $W$ of dimension $n$ such that the sum of the weights is 1 and $w_j \in [0, 1]$, then:

$$QOWA\,(a_1, a_2, \ldots, a_n) = g^{-1}\left( \sum_{j=1}^{n} w_j g\left(b_j\right) \right), \tag{3}$$

where $b_j$ is the $j^{\text{th}}$ largest of the $a_i$.

## 3. THE GENERALIZED HYBRID AVERAGING OPERATOR

The GHA operator is a generalization of the HA operator by using generalized means. It includes in the same formulation the weighted generalized mean and the GOWA operator. Then, this operator includes the WA, the OWA and the OWG operator as special cases. It is defined as follows.

**Definition 4.** A GHA operator of dimension $n$ is a mapping *GHA*: $R^n \rightarrow R$ that has an associated weighting vector $W$ of dimension $n$ with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, such that:

$$GHA\,(a_1, a_2, \ldots, a_n) = \left( \sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda}, \tag{4}$$

where $b_j$ is the $j^{\text{th}}$ largest of the $\hat{a}_i$ ($\hat{a}_i = n\omega_i a_i$, $i = 1,2,\ldots,n$), $\omega = (\omega_1, \ldots, \omega_n)^T$ is the weighting vector of the $a_i$, with $\omega_i \in [0, 1]$ and the sum of the weights is 1, and $\lambda$ is a parameter such that $\lambda \in (-\infty, \infty)$.

Note that if $\lambda \leq 0$, we can only use positive numbers $R^+$, in order to get consistent results. From a generalized perspective of the reordering step, we can distinguish between the descending GHA (DGHA) operator and the ascending GHA (AGHA) operator. Note that they can be used in situations where the highest value is the best result and in situations where the lowest value is the best result. But in a more efficient context, it is better to use one of them for one situation and the other one for the other situation. The weights of these operators are related by $w_j = w^*_{n-j+1}$, where $w_j$ is the $j$th weight of the DGHA and $w^*_{n-j+1}$ the $j$th weight of the AGHA operator. As we can see, the main difference is that in the AGHA operator, the elements $b_j$ ($j$= 1, 2, …, $n$) are ordered in an increasing way: $b_1 \leq b_2 \leq \ldots \leq b_n$ while in the DGHA (or GHA) they are ordered in a decreasing way.

If $B$ is a vector corresponding to the ordered arguments $b_j^{\lambda}$, we shall call this the ordered argument vector and $W^T$ is the transpose of the weighting vector, then, the GHA operator can be expressed as:

$$GHA\ (a_1, a_2, \ldots, a_n) = \left(W^T B\right)^{1/\lambda}. \tag{5}$$

Note that if the weighting vector is not normalized, i.e., $W = \sum_{j=1}^{n} w_j \neq 1$, then, the GHA operator can be expressed as:

$$GHA\ (a_1, a_2, \ldots, a_n) = \left(\frac{1}{W} \sum_{j=1}^{n} w_j b_j^{\lambda}\right)^{1/\lambda}. \tag{6}$$

The GHA operator is monotonic, commutative and idempotent. These properties can be proved with the following theorems.

**Theorem 1** (Monotonicity). Assume $f$ is the GHA operator, if $a_i \geq u_i$, for all $a_i$, then

$$f\,(a_1, a_2, \ldots, a_n) \geq f\,(u_1, u_2, \ldots, u_n). \tag{7}$$

**Proof.** Let

$$f\,(a_1, a_2, \ldots, a_n) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda}\right)^{1/\lambda}, \tag{8}$$

$$f\,(u_1, u_2, \ldots, u_n) = \left(\sum_{j=1}^{n} w_j v_j^{\lambda}\right)^{1/\lambda}. \tag{9}$$

Since $a_i \geq u_i$, for all $a_i$, it follows that, $b_i \geq v_i$, and then:

$$f\,(a_1, a_2, \ldots, a_n) \geq f\,(u_1, u_2, \ldots, u_n). \qquad \blacksquare$$

**Theorem 2** (Commutativity). Assume $f$ is the GHA operator, then

$$f\,(a_1, a_2, \ldots, a_n) = f\,(u_1, u_2, \ldots, u_n). \tag{10}$$

where $(u_1, u_2 \ldots, u_n)$ is any permutation of the arguments $(a_1, a_2 \ldots, a_n)$.

**Proof.** Let

$$f(a_1, a_2, \ldots, a_n) = \left( \sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda}, \tag{11}$$

$$f(u_1, u_2, \ldots, u_n) = \left( \sum_{j=1}^{n} w_j v_j^{\lambda} \right)^{1/\lambda}. \tag{12}$$

Since $(u_1, u_2, \ldots, u_n)$ is a permutation of $(a_1, a_2, \ldots, a_n)$, we have $b_j = v_j$, for all $j$, and then

$$f(a_1, a_2, \ldots, a_n) = f(u_1, u_2, \ldots, u_n).$$ ■

**Theorem 3** (Idempotency). Assume $f$ is the GHA operator, if $a_i = a$, for all $a_i$, then

$$f(a_1, a_2, \ldots, a_n) = a. \tag{13}$$

**Proof.** Since $a_i = a$, for all $a_i$, we have

$$f(a_1, a_2, \ldots, a_n) = \left( \sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda} = \left( \sum_{j=1}^{n} w_j a^{\lambda} \right)^{1/\lambda} = \left( a^{\lambda} \sum_{j=1}^{n} w_j \right)^{1/\lambda}. \tag{14}$$

Since $\sum_{j=1}^{n} w_j = 1$, we get $f(a_1, a_2, \ldots, a_n) = a$. ■

Note that this operator is not bounded by the maximum and the minimum because for some special situations it can be higher and lower than the maximum and the minimum, respectively. Mainly, this problem is found when using the hybrid maximum and minimum in the aggregation and in similar situations.

Another interesting issue to consider are the measures for characterizing the weighting vector $W = (w_1, w_2, \ldots, w_n)$ of the GHA operator such as the attitudinal character, the entropy of dispersion, the divergence of $W$ and the balance operator. Note that these measures follow the same methodology as the original version developed for the OWA operator (Yager, 1988; 1996b; 2002).

Using a similar methodology as it was used by Yager (2004) for the GOWA operator we can define the attitudinal character as follows:

$$\alpha(W) = \left( \sum_{j=1}^{n} w_j \left( \frac{n-j}{n-1} \right)^{\lambda} \right)^{1/\lambda}. \tag{15}$$

For the entropy of dispersion, we get:

$$H(W) = -\sum_{j=1}^{n} w_j \ln(w_j). \tag{16}$$

For the divergence of $W$:

$$DIV(W) = \sum_{j=1}^{n} w_j \left( \frac{n-j}{n-1} - \alpha(W) \right)^2. \tag{17}$$

And for the balance operator:

$$BAL\ (W) = \sum_{j=1}^{n} \left( \frac{n+1-2j}{n-1} \right) w_j \ . \tag{18}$$

Note that in this case, we could also distinguish between descending and ascending orders.

## 4. FAMILIES OF GHA OPERATORS

In the GHA operator we find different families of aggregation operators. Mainly, we can classify them in two types. The first type represents all the families found in the weighting vector $W$ and the second type, the families found in the parameter $\lambda$.

### 4.1 Analyzing the Weighting Vector *W*

By choosing a different manifestation of the weighting vector in the GHA operator, we are able to obtain different types of aggregation operators. For example, we can obtain the hybrid maximum, the hybrid minimum, the generalized mean (GM), the weighted generalized mean (WGM) and the GOWA operator.

The hybrid maximum is obtained if $w_1 = 1$ and $w_j = 0$, for all $j \neq 1$. The hybrid minimum is obtained if $w_n = 1$ and $w_j = 0$, for all $j \neq n$. More generally, if $w_k = 1$ and $w_j = 0$, for all $j \neq k$, we get for any $\lambda$, $GHA(a_1, a_2, \ldots, a_n) = b_k$, where $b_k$ is the $k$th largest argument $a_i$. The GM is found when $w_j = 1/n$, and $\omega_i = 1/n$, for all $a_i$. The WGM is obtained when $w_j = 1/n$, for all $a_i$. The GOWA is found when $\omega_i = 1/n$, for all $a_i$.

Following a similar methodology as it has been developed in Ahn, 2009; Ahn and Park, 2008; Emrouznejad, 2008; Liu, 2008; Merigó, 2008; Merigó and Casanovas, 2009; Merigó and Gil-Lafuente, 2009b; 2010; Wang and Parkan, 2007; Xu, 2005; Xu, 2008a; Yager, 1993; Yager, 1996a; Yager, 2003; Yager, 2007; Yager and Filev, 1994, we could study other particular cases of the GHA operator such as the step-GHA, the window-GHA, the olympic-GHA, the centered-GHA operator, the S-GHA operator, the median-GHA, the E-Z GHA, the maximal entropy GHA weights, the minimal variability GHA, the minimax disparity GHA weights, the nonmonotonic GHA operator, etc.

For example, when $w_{j*} = 1/m$ for $k \leq j^* \leq k + m - 1$ and $w_{j*} = 0$ for $j^* > k + m$ and $j^* < k$, we are using the window-GHA operator. Note that $k$ and $m$ must be positive integers such that $k + m - 1 \leq n$. Also note that if $m = k = 1$, the window-GHA is transformed in the hybrid maximum. If $m = 1$, $k = n$, the window-GHA becomes the hybrid minimum. And if $m = n$ and $k = 1$, the window-GHA is transformed in the GM.

The olympic-GHA, based on the olympic average (Yager, 1993), is found when $w_1 = w_n = 0$, and for all others $w_{j*} = 1/(n - 2)$. Note that if $n = 3$ or $n = 4$, the olympic-GHA is transformed in the median-GHA and if $m = n - 2$ and $k = 2$, the window-GHA is transformed in the olympic-GHA.

Note that the median can also be used as GHA operators. For the median-GHA, if $n$ is odd we assign $w_{(n+1)/2} = 1$ and $w_{j*} = 0$ for all others. If $n$ is even we assign for example, $w_{n/2} = w_{(n/2)+1} = 0.5$ and $w_{j*} = 0$ for all others.

For the weighted median-GHA, we select the argument $b_k$ that has the $k$th largest argument such that the sum of the weights from 1 to $k$ is equal or higher than 0.5 and the sum of the weights from 1 to $k - 1$ is less than 0.5.

Another type of aggregation that could be used is the E-Z GHA weights that it is based on the E-Z OWA weights (Yager, 2003). In this case, we should distinguish between two classes. In the first class, we assign $w_{j*} = (1/q)$ for $j* = 1$ to $q$ and $w_{j*} = 0$ for $j* > q$, and in the second class, we assign $w_{j*} = 0$ for $j* = 1$ to $n - q$ and $w_{j*} = (1/q)$ for $j* = n - q + 1$ to $n$. If $q = 1$ for the first class, the E-Z GHA becomes the hybrid maximum. And if $q = 1$ for the second class, the E-Z GHA becomes the hybrid minimum.

A further interesting family is the S-GHA operator. It can be subdivided in three classes: the "orlike", the "andlike" and the generalized S-GHA operator. The generalized S-GHA operator is obtained when $w_1 = (1/n)(1 - (\alpha + \beta)) + \alpha$, $w_n = (1/n)(1 - (\alpha + \beta)) + \beta$, and $w_j = (1/n)(1 - (\alpha + \beta))$ for $j = 2$ to $n - 1$ where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$. Note that if $\alpha = 0$, the generalized S-GHA operator becomes the "andlike" S-GHA operator and if $\beta = 0$, it becomes the "orlike" S-GHA operator. Also note that if $\alpha + \beta = 1$, we get the generalized hybrid Hurwicz criteria.

Another family of aggregation operator that could be used is the centered-GHA operator. Following the same methodology than Yager, 2007, we could define a GHA operator as a centered aggregation operator if it is symmetric, strongly decaying and inclusive. Note that these properties have to be accomplished for the weighting vector $w$ of the OWA operator but not necessarily for the weighting vector $\omega$ of the WA. It is symmetric if $w_j = w_{j+n-1}$. It is strongly decaying when $i < j \leq (n + 1)/2$ then $w_i < w_j$ and when $i > j \geq (n + 1)/2$ then $w_i < w_j$. It is inclusive if $w_j > 0$. Note that it is possible to consider a softening of the second condition by using $w_i \leq w_j$ instead of $w_i < w_j$. We shall refer to this as softly decaying centered-GHA operator. Another particular situation of the centered-GHA operator appears if we remove the third condition. We shall refer to it as a non-inclusive centered-GHA operator.

### 4.2 Analyzing the Parameter λ

If we analyze different values of the parameter $\lambda$, we obtain another group of particular cases such as the usual HA operator, the hybrid geometric averaging (HGA) operator, the hybrid harmonic averaging (HHA) operator and the hybrid quadratic averaging (HQA) operator.

When $\lambda = 1$, we get the HA operator.

$$HA\ (a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j \,. \tag{19}$$

Note that if $w_j$ = 1/$n$, for all $a_i$, we get the WA and if $\omega_j$ = 1/$n$, for all $a_i$, we get the OWA operator. If $w_j$ = 1/$n$, and $\omega_j$ = 1/$n$, for all $a_i$, then, we get the arithmetic mean (AM). From a generalized perspective of the reordering step we can distinguish between the DHA operator and the AHA operator.

When $\lambda = 0$, the GHA operator becomes the HGA operator.

$$HGA\ (a_1, a_2, \ldots, a_n) = \prod_{j=1}^{n} b_j^{w_j} \,. \tag{20}$$

If $w_j$ = 1/$n$, for all $a_i$, we get the WGM and if $\omega_j$ = 1/$n$, for all $a_i$, we get the OWG operator. If $w_j$ = 1/$n$, and $\omega_j$ = 1/$n$, for all $a_i$, then, we get the geometric mean (GM). In this case, we can also distinguish between descending (DHGA) and ascending (AHGA).

When $\lambda = -1$, we get the HHA operator.

$$HHA\ (a_1, a_2, \ldots, a_n) = \frac{1}{\sum_{j=1}^{n} \frac{w_j}{b_j}} \,. \tag{21}$$

In this case, we get the descending HHA (DHHA) operator and the ascending HHA (AHHA) operator. Note that if $w_j$ = 1/$n$, for all $a_i$, we get the weighted harmonic mean (WHM) and if $\omega_j$ = 1/$n$, for all $a_i$, we get the ordered weighted harmonic averaging (OWHA) operator. If $w_j$ = 1/$n$, and $\omega_j$ = 1/$n$, for all $a_i$, then, we get the harmonic mean (HM).

When $\lambda = 2$, we get the HQA operator.

$$HQA\ (a_1, a_2, \ldots, a_n) = \left( \sum_{j=1}^{n} w_j b_j^2 \right)^{1/2} \,. \tag{22}$$

In this case, we can also distinguish between the descending HQA (DHQA) operator and the ascending HQA (AHQA) operator. If $w_j$ = 1/$n$, for all $a_i$, we get the WQM and if $\omega_j$ = 1/$n$, for all $a_i$, we get the OWQA operator. If $w_j$ = 1/$n$, and $\omega_j$ = 1/$n$, for all $a_i$, then, we get the quadratic mean (QM).

Note that we could analyze other families by using different values in the parameter $\lambda$. Also note that it is possible to study these families individually. Then, we could develop for each case, a similar analysis as it has been developed in Sections 3 and 4.1, where we study different properties and families of the aggregation operator.

## 5. THE QUASI-HA OPERATOR

Going a step further, it is possible to generalize the GHA operator by using quasi-arithmetic means in a similar way as it was done for the GOWA operator (Beliakov, 2005). The result is the Quasi-HA operator which is a hybrid version of the Quasi-OWA operator (Fodor *et al*., 1995). It can be defined as follows.

**Definition 4.** A Quasi-HA operator of dimension $n$ is a mapping *Quasi-HA*: $R^n \rightarrow R$ that has an associated weighting vector $W$ of dimension $n$ with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, such that:

$$\text{Quasi-HA}\ (a_1, \ldots, a_n) = g^{-1}\left(\sum_{j=1}^{n} w_j g\left(b_j\right)\right), \tag{23}$$

where $b_j$ is the $j^{\text{th}}$ largest of the $\hat{a}_i$ ($\hat{a}_i = n\omega_i a_i$, $i = 1,2,\ldots,n$), $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weighting vector of the $a_i$, with $\omega_i \in [0, 1]$ and the sum of the weights is 1.

As we can see, we replace $b^{\lambda}$ with a general continuous strictly monotone function $g(b)$. In this case, the weights of the ascending and descending versions are also related by $w_j = w^*_{n-j+1}$, where $w_j$ is the $j$th weight of the Quasi-DHA and $w^*_{n-j+1}$ the $j$th weight of the Quasi-AHA operator.

Note that all the properties and particular cases commented in the GHA operator, are also included in this generalization. For example, we could study different families of Quasi-HA operators such as the Quasi-OWA, the Quasi-WA, the Quasi-step-HA, the Quasi-window-HA, the Quasi-median-HA, the Quasi-olympic-HA, the Quasi-centered-HA, etc.

Another interesting issue to consider is the attitudinal character of the Quasi-HA operator. Following a similar methodology than Beliakov, 2005, we can define the following measure:

$$\alpha(W) = g^{-1}\left(\sum_{j=1}^{n} w_j g\left(\frac{n-j}{n-1}\right)\right). \tag{24}$$

Note that in this case it is also possible to consider other measures such as the entropy of dispersion, the divergence of $W$ or the balance operator. Their formulation is practically the same as it has been explained in the end of Section 3 for the GHA operator.

A further interesting aspect is that the Quasi-HA operator includes a lot of other particular cases that are not included in the GHA operator. For example, we could mention the trigonometric HA operator, the exponential HA operator and the radical HA operator.

The trigonometric HA is found when $g_1(t) = \sin((\pi/2)\ t)$, $g_2(t) = \cos((\pi/2)\ t)$ and $g_3(t) = \tan((\pi/2)\ t)$ are the generating functions. Then, the trigonometric HA functions are:

$$HA\ (a_1, \ldots, a_n) = \frac{2}{\pi}\arcsin\left(\sum_{j=1}^{n} w_j \sin\left(\frac{\pi}{2} b_j\right)\right), \tag{25}$$

$$HA\ (a_1, \ldots, a_n) = \frac{2}{\pi}\arccos\left(\sum_{j=1}^{n} w_j \cos\left(\frac{\pi}{2} b_j\right)\right), \tag{26}$$

$$HA\ (a_1, \ldots, a_n) = \frac{2}{\pi}\arctan\left(\sum_{j=1}^{n} w_j \tan\left(\frac{\pi}{2} b_j\right)\right). \tag{27}$$

The exponential HA is found when $g(t) = \gamma^t$ , if $\gamma \neq 1$, and $g(t) = t$, if $\gamma = 1$. Then, the exponential HA operator is: $\log_\gamma\left(\sum_{j=1}^{n} w_j \gamma^{b_j}\right)$, if $\gamma \neq 1$; and the HA if $\gamma = 1$.

The radical HA is found if $\gamma > 0$, $\gamma \neq 1$, and the generating function is $g(t) = \gamma^{1/t}$. Then, the radical IOWA operator is:

$$HA\ (a_1, \ldots, a_n) = \left(\log_\gamma\left(\sum_{j=1}^{n} w_j \gamma^{1/b_j}\right)\right)^{-1}. \tag{28}$$

Finally, note that in these cases it is also possible to study their properties and different particular cases as it has been explained in Sections 3 and 4.1.

## 6. NUMERICAL EXAMPLE

Now, we are going to develop an application of the new approach in a decision making problem. We will analyze an investment selection problem where an investor is looking for an optimal investment. Note that other decision making applications could be developed (Alonso *et al*., 09; Herrera *et al*., 2003; Xu, 2008b) such as the selection of financial products, the selection of strategies, the selection of human resources, etc.

We will develop the analysis considering a wide range of particular cases of the GHA operator such as the maximum, the minimum, the arithmetic mean (AM), the WA, the OWA, the OWQA, the HA, the AHA, the HQA and the HGA. Note that we do not consider the hybrid maximum and the hybrid minimum because sometimes its results are inconsistent. This inconsistency happens because the results may be higher than the maximum and lower than the minimum. Due to this, we will not use them in this example. The hybrid maximum and minimum are useful for taking decisions but they do not correctly aggregate the information in the sense that they are not always bounded by the maximum and minimum arguments.

Assume an investor wants to invest some money in an enterprise in order to obtain high benefits. Initially, he considers five possible alternatives.

- $A_1$ is a computer company.
- $A_2$ is a food company.
- $A_3$ is a TV company.
- $A_4$ is a chemical company.
- $A_5$ is a car company.

In order to evaluate these investments, the investor uses a group of experts. This group of experts considers that the key factor is the economic environment of the economy. After detailed analysis, they consider five possible situations for the economic environment: $S_1$ = Very bad, $S_2$ = Bad, $S_3$ = Normal, $S_4$ = Good, $S_5$ = Very good. The expected results depending on the state of nature $S_i$ and the alternative $A_k$ are shown in Table 1.

Table 1. Payoff matrix

| | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ |
|---|---|---|---|---|---|
| $A_1$ | 30 | 60 | 50 | 80 | 20 |
| $A_2$ | 30 | 30 | 90 | 60 | 40 |
| $A_3$ | 70 | 40 | 50 | 20 | 60 |
| $A_4$ | 50 | 70 | 30 | 40 | 50 |
| $A_5$ | 90 | 10 | 10 | 70 | 70 |

In this example, we assume that the group of experts assumes the following weighting vector for all the cases of the WA and the OWA operator: $W = (0.1, 0.1, 0.2, 0.3, 0.3)$. With this information, it is possible to aggregate it in order to take a decision. First, we consider the results obtained with some basic aggregation operators. The results are shown in Table 2.

Table 2. Aggregated results 1

| | Max | Min | AM | WA | OWA |
|---|---|---|---|---|---|
| $A_1$ | 80 | 20 | 48 | 49 | 39 |
| $A_2$ | 90 | 30 | 50 | 54 | 44 |
| $A_3$ | 70 | 20 | 48 | 45 | 41 |
| $A_4$ | 70 | 30 | 48 | 45 | 40 |
| $A_5$ | 90 | 10 | 50 | 54 | 36 |

As we can see, the optimal investment is different depending on the aggregation operator used. In the following, we consider other particular cases of the GHA operator with more complexity. The results are shown in Table 3.

Table 3. Aggregated results 2

| | OWQA | HA | AHA | HQA | HGA |
|---|---|---|---|---|---|
| $A_1$ | 43.4 | 36.5 | 61.5 | 46.9 | 29.4 |
| $A_2$ | 45.0 | 39 | 69 | 49.7 | 28.3 |
| $A_3$ | 44.1 | 36 | 54 | 41.1 | 32.1 |
| $A_4$ | 44.6 | 37 | 53 | 40.3 | 34.4 |
| $A_5$ | 48.3 | 34.5 | 73.5 | 51.4 | 17.5 |

Again, we can see that the optimal investment is not the same for all the aggregations used. Note that other types of GHA operators may be used in the analysis such as the ones explained in Section 4. Note that the selection of the particular type of GHA operator in the decision process will depend on the particular interests of the decision maker in the specific problem considered. For example, if the decision maker is optimistic, he will go for a particular case of the GHA close to the maximum (or optimistic criteria) and if he is pessimistic, he will go to a specific case close to the minimum. A further interesting issue is to establish an ordering of the investments.

This is very useful when the investor wants to consider more than one alternative. The results are shown in Table 4.

Table 4: Ordering of the investments

| | Ordering | | Ordering |
|---|---|---|---|
| Max | $A_5 \succ A_3 \succ A_4 \succ A_1 = A_2$ | OWQA | $A_5 \succ A_2 \succ A_4 \succ A_3 \succ A_1$ |
| Min | $A_2 = A_4 \succ A_1 = A_3 \succ A_5$ | HA | $A_2 \succ A_4 \succ A_1 \succ A_3 \succ A_5$ |
| AM | $A_2 = A_5 \succ A_1 = A_3 = A_4$ | AHA | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| WA | $A_2 = A_5 \succ A_1 \succ A_3 = A_4$ | HQA | $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$ |
| OWA | $A_2 \succ A_3 \succ A_4 \succ A_1 \succ A_5$ | HGA | $A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$ |

As we can see, we get different orderings of the investments depending on the aggregation operator used.

## 7. CONCLUSIONS

We introduced a new type of aggregation operator: the generalized hybrid averaging (GHA) operator. It is a generalization of the hybrid averaging (HA) operator by using generalized means. We saw that it is very useful when we want to consider subjective probabilities and the attitudinal character of the decision maker in the same problem. With this generalization we found different special cases such as the hybrid geometric averaging (HGA), the hybrid quadratic averaging (HQA), the WA, the OWA operator, the WGM, the OWG operator, the WQM, the OWQA operator, etc. We further generalized the GHA operator by using quasi-arithmetic means, obtaining the quasi-HA operator.

We ended the paper with an application of the new approach in a decision making problem. We focussed on a financial problem where we saw the usefulness of the new approach in the selection of investments. The main advantage of using the GHA operator is that it gives a complete view of the decision problem because it includes a lot of particular cases that can be used in the aggregation of the information according to the interests of the decision maker.

In future research, we expect to develop further extensions of the GHA operator by adding new characteristics in the problem such as the use of order inducing variables, interval numbers, fuzzy numbers, linguistic variables, etc. We will also consider other decision making applications.


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## REFERENCES


Ahn, B.S. (2009) “Some remarks on the LSOWA approach for obtaining OWA operator weights”, *International Journal of Intelligent Systems*, 24, 1265–1279.

Ahn, B.S. and H. Park (2008) “Least-squared ordered weighted averaging operator weights”, *International Journal of Intelligent Systems*, 23, 33–49.

Alonso, S., Cabrerizo, F.J., Chiclana, F., Herrera, F. and E. Herrera-Viedma (2009) “Group decision making with incomplete fuzzy linguistic preference relations”, *International Journal of Intelligent Systems*, 24, 201–222.

Beliakov, G. (2005) “Learning Weights in the Generalized OWA Operators”, *Fuzzy Optimization and Decision Making*, 4, 119–130.

Beliakov, G., Pradera, A. and T. Calvo (2007) *Aggregation Functions: A guide for practitioners*, Springer-Verlag, Berlin.

Calvo, T., Mayor, G. and R. Mesiar (2002) *Aggregation Operators: New Trends and Applications*, Physica-Verlag, New York.

Canós, L. and V. Liern (2008) “Soft computing-based aggregation methods for human resource management”, *European Journal of Operational Research*, 189, 669–681.

Dyckhoff, H. and W. Pedrycz (1984) “Generalized means as model of compensative connectives”, *Fuzzy Sets and Systems*, 14, 143–154.

Emrouznejad, A. (2008) “MP-OWA: The most preferred OWA operator”, *Knowledge-Based Systems*, 21, 847–851.

Fodor, J., Marichal, J.L. and M. Roubens (1995) “Characterization of the ordered weighted averaging operators”, *IEEE Transactions on Fuzzy Systems*, 3, 236–240.

Herrera, F., Herrera-Viedma, E. and F. Chiclana (2003) “A study of the origin and uses of the ordered weighted geometric operator in multicriteria decision making”, *International Journal of Intelligent Systems*, 18, 689–707.

Karayiannis, N. (2000) “Soft learning vector quantization and clustering algorithms based on ordered weighted aggregation operators”, *IEEE Transactions on Neural Networks*, 11, 1093–1105.

Liu, X. (2008) “A general model of parameterized OWA aggregation with given orness level”, *International Journal of Approximate Reasoning*, 48, 598–627.

Merigó, J.M. (2008) *Nuevas extensiones a los operadores OWA y su aplicación en los métodos de decisión*, PhD Thesis (In Spanish), Department of Business Administration, University of Barcelona.

Merigó, J.M. and M. Casanovas (2009) “Induced aggregation operators in decision making with the Dempster-Shafer belief structure”, *International Journal of Intelligent Systems*, 24, 934–954.

Merigó, J.M. and M. Casanovas (2010) “The uncertain generalized OWA operator and its application in financial decision making”, *International Journal of Information Technology and Decision Making*, 9, (forthcoming).

Merigó, J.M. and A.M. Gil-Lafuente (2008) “The generalized adequacy coefficient and its application in strategic decision making”, *Fuzzy Economic Review*, 13, 17–36.

Merigó, J.M. and A.M. Gil-Lafuente (2009a) “OWA operators in generalized distances”, *International Journal of Applied Mathematics and Computer Science*, 5, 11–18.

Merigó, J.M. and A.M. Gil-Lafuente (2009b) "The induced generalized OWA operator", *Information Sciences*, 179, 729–741.

Merigó, J.M. and A.M. Gil-Lafuente (2010) "New decision-making techniques and their application in the selection of financial products", *Information Sciences*, 180, 2085–2094.

Wang, Y.M. and C. Parkan (2007) "A preemptive goal programming method for aggregating OWA operator weights in group decision making", *Information Sciences*, 177, 1867–1877.

Wei, G.W. (2009) "Uncertain linguistic hybrid geometric mean operator and its application to group decision making under uncertain linguistic environment". *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 17, 251–267.

Xu, Z.S. (2004) "Uncertain linguistic aggregation operators based approach to multiple attribute group decision making under uncertain linguistic environment", *Information Sciences*, 168, 171–184.

Xu, Z.S. (2005) "An overview of methods for determining OWA weights", *International Journal of Intelligent Systems*, 20, 843–865.

Xu, Z.S. (2008a) "Dependent uncertain ordered weighted averaging operators", *Information Fusion*, 9, 310–316.

Xu, Z.S. (2008b) "On multi-period multi-attribute decision making", *Knowledge-Based Systems*, 21, 164–171.

Xu, Z.S. (2009) Fuzzy harmonic mean operators. *International Journal of Intelligent Systems*, 24, 152–172.

Xu, Z.S. and Q.L. Da (2002) "The uncertain OWA operator", *International Journal of Intelligent Systems*, 17, 569–575.

Xu, Z.S. and Q.L. Da (2003) "An overview of operators for aggregating the information", *International Journal of Intelligent Systems*, 18, 953–969.

Yager, R.R. (1988) "On ordered weighted averaging aggregation operators in multi-criteria decision making", *IEEE Transactions on Systems, Man and Cybernetics B*, 18, 183–190.

Yager, R.R. (1993) "Families of OWA operators", *Fuzzy Sets and Systems*, 59, 125–148.

Yager, R.R. (1996a) "Quantifier guided aggregation using OWA operators", *International Journal of Intelligent Systems*, 11, 49–73.

Yager, R.R. (1996b) "Constrained OWA aggregation", *Fuzzy Sets and Systems*, 81, 89–101.

Yager, R.R. (2002) "Heavy OWA operators", *Fuzzy Optimization and Decision Making*, 1, 379–397.

Yager, R.R. (2003) "E-Z OWA weights", in: *Proc. 10th IFSA World Congress*, Istanbul, Turkey, 39–42.

Yager, R.R. (2004) "Generalized OWA aggregation operators", *Fuzzy Optimization and Decision Making*, 3, 93–107.

Yager, R.R. (2007) "Centered OWA operators", *Soft Computing*, 11, 631–639.

Yager, R.R. (2008) "Using trapezoids for representing granular objects: Applications to learning and OWA aggregation", *Information Sciences*, 178, 363–380.

Yager, R.R. and D.P. Filev (1994) "Parameterized andlike and orlike OWA operators", *International Journal of General Systems*, 22, 297–316.

Yager, R.R. and J. Kacprzyk (1997) *The Ordered Weighted Averaging Operators: Theory and Applications*, Kluwer Academic Publishers, Norwell, MA.

Zhao, H., Xu, Z.S., Ni, M. and S. Liu (2010) "Generalized aggregation operators for intuitionistic fuzzy sets", *International Journal of Intelligent Systems*, 25, 1–30.