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Qualitative Analysis of the Goodwin Model of the Growth Cycle

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ABSTRACT

Goodwin's model is a set of ordinary differential equations and is a well-known model of the growth cycle. However, its four constants require an extensive numerical study of its two differential equations to identify all possible unsteady state behaviors, i.e. phase portraits, which corresponds to infinitely many combinations of numerical values of the constants. Qualitative interpretation of Goodwin's model solves these problems by replacing all numerical constants and all derivatives by trends (increasing, constant and decreasing). The model has two variables — the employment rate V, and the labour share U. A solution of the qualitative Goodwin's model is a scenario. An example of a Goodwin's scenario is — V *is increasing more and more rapidly,* U *is decreasing and the decrease is slowing down.* The complete set of all possible 41 Goodwin's scenarios and 168 time transitions among them are given. This result qualitatively represents all possible unsteady state Goodwin's behaviours. It is therefore possible to predict all possible future behaviours if a current behaviour is known/chosen. A prediction example is presented in details. No prior knowledge of qualitative model theory is required.

Keywords: Goodwin model; business cycle; qualitative; scenario; transition.
JEL classification: E17.
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Análisis cualitativo del modelo de Goodwin de ciclos de crecimiento

RESUMEN

El modelo de Goodwin es un conjunto de ecuaciones diferenciales ordinarias y resulta un modelo bien conocido para ciclos de crecimiento. Sin embargo, sus cuatro constantes requieren de un extenso estudio numérico de sus dos ecuaciones diferenciales para identificar todos los posibles comportamientos de estado no estacionario, i.e. retratos de fase, que corresponden a infinitamente muchas combinaciones de valores numéricos de las constantes. La interpretación cualitativa del modelo de Goodwin resuelve estos problemas reemplazando todas las constantes numéricas y todas las derivadas por tendencias (creciente, constante y decreciente). El modelo consiste en dos variables: la tasa de empleabilidad V y la repartición del valor agregado U. Una solución del modelo cualitativo de Goodwin es un escenario. Un ejemplo de escenario de Goodwin es el siguiente: *V es creciente cada vez más rápidamente y U es decreciente pero el decrecimiento se está ralentizando*. Se obtiene el conjunto completo de los 41 escenarios posibles de Goodwing con las 168 transiciones temporales entre ellas. Este resultado representa cualitativamente todos los posibles comportamientos de estado no estacionario de Goodwin. Por tanto, es posible predecir todos los comportamientos futuros posibles si un comportamiento actual es conocido o elegido. Un ejemplo de predicción es presentado en detalle. No se requiere ningún conocimiento previo de la teoría de modelos cualitativos.

Palabras claves: modelo de Goodwin; ciclo empresarial; cualitativo; escenario; transición.
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1. Introduction

Goodwin (1967) states a simple but useful model of the struggle between capital and labour for shares in national income, based on the classic Volterra–Lotka predator–prey model for fish populations. Since then, the model has been extended in many directions and as such has proved to be a useful framework for combining growth and cycles in a simple non-linear model (Desai *et al.*, 2006).

Some researchers intended to add additional variables to the original model in order to make the model more generalized and realistic, see e.g. Sordi and Vercelli, (2014) or Sportelli, (1995). Other researchers studied the stability and other properties of the original model (Yoshida and Asada, 2007; Cao and Jiang, 2011; Veneziani and Mohun, 2006). Those who tried to evaluate the model empirically (Weber, 2005; Harvie, 2000; Moura Jr. and Ribeiro, 2013) and investigate the cyclical behaviour, used the data available from different sources or just solved the model numerically.

However, a numerical solution of Goodwin's model requires knowledge of all the four numerical constants. A qualitative interpretation of such parameters is possible and is studied in this paper. The paper presents a qualitative approach to the Goodwin model. Deep knowledge items reflect undisputed elements of the corresponding theory. The law of gravity is an example. This law has no exceptions. This is a typical feature of deep-knowledge items. Soft sciences as e.g. macroeconomics, are just very rarely based on deep-knowledge items. Goodwin's model is based partially on shallow knowledge.

A shallow-knowledge item is usually a heuristic or a result of a statistical analysis of observations and has usually many exceptions, see e.g. Oliveira and Rezende, (2013) or Ahn and Kim (2009).

2. Qualitative algebra

The following description of Qualitative Algebra is sufficient to make this paper self-contained. For additional details see, e.g. Dohnal (1991), Kuipers, (1994), Salles *et al.* (2006) or Šuc *et al.* (2004).

Volterra-Lotka model is a set of ordinary nonlinear differential equations (ONDEs). A qualitative solution of ONDEs is specified if all its *n* qualitative variables:

$$x_1, x_2, \ldots, x_n \tag{1}$$

are described by the corresponding qualitative triplets:

$$(X_1, DX_1, DDX_1), (X_2, DX_2, DDX_2), \ldots (X_n, DX_n, DDX_n), \tag{2}$$

where X_i is the i-th variable and DX_i and DDX_i are the first and second qualitative derivatives with respect to the independent variable t (which is usually time). DX is the qualitative interpretation of a numerical value for dx/dt. The reason why the third and higher derivatives are ignored is that these derivatives are known just rarely.

A qualitative model has m qualitative solutions i.e. scenarios. The j-th qualitative state is the n-triplet:

$$(X_1, DX_1, DDX_1), (X_2, DX_2, DDX_2), \ldots (X_n, DX_n, DDX_n)_j, \tag{3}$$

where $j = 1,2,\ldots,m.$

A simple algorithm can evaluate all possible transitions among the set of one-dimensional scenarios $n = 1$ (see (3)). One-dimensional transitions are based on the list of possible one-dimensional transitions, see Table 1. Multidimensional transitions must satisfy the Table 1 for n one-dimensional scenarios. However, this table is not a dogma. If a user feels/ knows/believes that a certain transition is not possible then this transition can simply be removed from the table.

Table 1: Some transition rules

	From		To	Or	Or	Or	Or	Or	Or
			a	b	c	d	e	f	g
1	+ + +	→	+ + 0						
2	+ + 0	→	+ + +	+ + -					
3	+ + -	→	+ + 0	+ 0 -	+ 0 0				
4	+ 0 +	→	+ + +						
5	+ 0 0	→	+ + +	+ - -					
6	+ 0 -	→	+ - -						
7	+ - +	→	+ - 0	+ 0 +	+ 0 0	0 - +	0 0 +	0 0 0	0 - 0
8	+ - 0	→	+ - +	+ - -	0 - 0				
9	+ - -	→	+ - 0	0 - -	0 - 0				

An oriented graph is commonly used to represent graphically the set of all the transitions. If it is possible to transfer the r-th solution (3) into the s-th solution, then an oriented arc represents the corresponding transition from the node r to the node s.

A qualitative addition

$$X_i + X_j = X_s \tag{4}$$

is represented by the matrix shown in Table 2.

Table 2: Qualitative Addition

+		X_j		
		+	o	-
X_i	+	+	+	?
	o	+	o	-
	-	?	-	-

It is sometimes possible to find more than one qualitative value. It is impossible to predict a sign of the result:

$$(X_i\textit{=positive}) + (X_j\textit{=negative}) = (X_s\textit{=?}) \tag{5}$$

A qualitative derivative of a sum of qualitative variables is a sum of their qualitative derivatives.

$$DX_i + DX_j = DX_s$$
$$DDX_i + DDX_j = DDX_s \tag{6}$$

A qualitative multiplication

$$X_i * X_j = X_s \tag{7}$$

is described by Table 3.

Table 3: Qualitative Multiplication

*		X_j		
		+	o	-
X_i	+	+	o	-
	o	o	o	o
	-	-	o	+

A known relation for the first qualitative derivative gives

$$X_i * DX_i + X_j * DX_j = DX_s \tag{8}$$

ONDEs are interpreted as a set of qualitative differential equations and solved using qualitative additions and multiplications.

A multiplication by a qualitative constant *c* is irrelevant:

$$c \times X = 2 \times X = 2000 \times X = X \tag{9}$$

where c is a numerical constant and X is variable.

3. The qualitative Goodwin model of the growth cycle

The Goodwin model can be represented as follows (Desai *et al.*, 2006; Sordi and Vercelli, 2014):

$$\begin{aligned} \frac{\dot{v}}{v} &= \frac{1-u}{\sigma} - (\alpha + \beta) \\ \frac{\dot{u}}{u} &= -(\gamma + \alpha) + \rho v \end{aligned} \tag{10}$$

where numerical constants α, δ, γ and σ are considered to be positive; v is the employment rate and u is the labour share.

A numerical solution of the set of differential equations shown in (10) requires knowledge of all the four constants. Therefore a qualitative interpretation of the model represented by (10) generates a meaningful solution based on trends only.

A qualitative interpretation of Eq. (10) is, see (9):

$$\begin{aligned} DV + V \times U &= V \\ DU + U &= V \times U \end{aligned} \tag{11}$$

Algorithms used to solve qualitative models are combinatorial tasks and are not studied in this paper; for details see e.g. Dohnal (1991). Table 4 gives 41 scenarios, see (3), of the qualitative Goodwin model described by (10).

Table 4: List of all scenarios of Goodwin model

No	V	U	No	V	U
1	+ + +	+ + +	22	+ 0 -	+ + +
2	+ + +	+ + 0	23	+ 0 -	+ + 0
3	+ + +	+ + -	24	+ 0 -	+ + -
4	+ + +	+ 0 +	25	+ - +	+ + -
5	+ + +	+ - +	26	+ - +	+ 0 -
6	+ + +	+ - 0	27	+ - +	+ - +
7	+ + +	+ - -	28	+ - +	+ - 0
8	+ + 0	+ + +	29	+ - +	+ - -
9	+ + 0	+ + 0	30	+ - 0	+ + -
10	+ + 0	+ + -	31	+ - 0	+ 0 -
11	+ + 0	+ 0 +	32	+ - 0	+ - +
12	+ + 0	+ - +	33	+ - 0	+ - 0
13	+ + -	+ + +	34	+ - 0	+ - -
14	+ + -	+ + 0	35	+ - -	+ + +
15	+ + -	+ + -	36	+ - -	+ + 0
16	+ + -	+ 0 +	37	+ - -	+ + -
17	+ + -	+ - +	38	+ - -	+ 0 -
18	+ 0 +	+ - +	39	+ - -	+ - +
19	+ 0 +	+ - 0	40	+ - -	+ - 0
20	+ 0 +	+ - -	41	+ - -	+ - -
21	+ 0 0	+ 0 0			

There are 168 transitions among them. As an example, a set of 20 transitions among the scenarios included in Table 4, is given in Table 5.

Table 5: Some transitions of Goodwin model

No of transition	From	To
1	1	2
2	1	8
3	1	9
4	2	1
5	2	3
6	2	8
7	2	9
8	2	10
9	3	2
16	5	11
26	8	14
41	11	8
61	14	24
78	19	5
83	21	1
94	24	38
109	28	19
136	33	28
145	34	33
156	38	34

Weber (2005) simulates the behaviour of Goodwin's model by applying randomly selected values. As a result, a graph is achieved, which represents the cyclical behaviour of the model. One-dimensional transitions, given in Table 1, are used to identify some possible transitions among the scenarios set, see Table 4.

Let us suppose that the current scenario is the scenario No. 28. The following time sequence of scenarios is possible, see Table 4:

$$28 \xrightarrow{109} 19 \xrightarrow{78} 5 \xrightarrow{16} 11 \xrightarrow{41} 8 \xrightarrow{26} 14 \xrightarrow{61} 24 \xrightarrow{94} 38 \xrightarrow{156} 34 \xrightarrow{145} 33 \xrightarrow{136} 28 \qquad (12)$$

The time sequence shown in (12) corresponds to the cyclical behaviour of the model in Weber (2005). The described behaviour is not the only possible one, keeping in mind that there are 168 possible transitions among 41 scenarios.

The following examples give an interpretation of qualitative Goodwin's results, see Tables 4 and 5. Different qualitative answers are presented. The first qualitative question to be answered is the following: Is there a steady state?; i.e. is there the following scenario (see n = 2 in (2)?:

V	U
+ 0 0	+ 0 0

The answer is solved just by searching through Table 4. According to this table, scenario No. 21 is the qualitative steady state as both variables V and U are positive and have the first and the second derivatives equal to zero.

Scenario No. 1 (see Table 4) has the following triplets:

V	U
+ + +	+ + +

It means that both employment rate and labour share are increasing more and more rapidly. Scenario No. 21 is the steady state. The list of transitions (see Table 5) indicates that it is possible to transfer Scenario No. 21 into Scenario No. 1. It is impossible to transfer Scenario No.1 into Scenario No. 21. For instance, if it is necessary to reach the steady state of the system, there are many paths being available. For example:

$$28 \xrightarrow{109} 19 \xrightarrow{78} 5 \xrightarrow{16} 11 \xrightarrow{41} 8 \xrightarrow{26} 14 \xrightarrow{58} 15 \xrightarrow{65} 21 \qquad (13)$$

4. Conclusion

The well-known Goodwin's model is the first model which tried to combine cyclical behaviour and economic growth. Therefore a complex system with its negative features must be studied. It means that business cycle analysis must be done under the following conditions:

- Severe shortage of information
- High level of subjectivity of available knowledge
- Inconsistencies of information items of interdisciplinary nature

Throughout the years the model has been tested both theoretically and numerically. Although it was mentioned above that constants α, δ, γ and σ of the system (10) are hard to identify numerically, so this paper attempted to evaluate the model qualitatively and apply the achieved results into practice.

We have qualitatively described the trajectory of the cyclical behaviour of the Goodwin's model, studied in Weber (2005), by a sequence of one-dimensional scenarios and found out that this sequence represents only a subset of our qualitative model.

Our qualitative Goodwin model gives all the possible developments based on two variables. No statistical data sets are needed and all the possible solutions are identified.

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