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UP METHODS IN THE ALLOCATION OF INDIVISIBILITIES WHEN PREFERENCES ARE SINGLE-PEAKED*

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Abstract

We consider allocation problems with indivisible goods when agents' preferences are single-peaked. We propose natural rules (called *up methods*) to solve such a class of problems. We analyzed the properties those methods satisfy and we provide a characterization of them. We also prove that these methods can be interpreted as extensions to the indivisible case of the so-called *equal-distance rule*.

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1 Introduction

Imagine that we face up the problem of allocating shifts among doctors at a health center, when each doctor has *single-peaked* preferences over worked hours. This means that she has a most preferred amount of hours to work. And, if she has to work more than this preferred amount, the less the better. Similarly, if she has to work less than this preferred amount the more the better. We find analogous situation in the allotment of crew members to flights or teaching hours to faculty members, among others. The above situation belongs to a general class of problems called *allocation problems with indivisibilities when preferences are single-peaked*.

We propose here a sequential procedure to solve allocation problems. It is based on accommodating the task unit by unit. At each step of the process we need to decide the agent getting the unit. To do that we use a sort of order, called *monotonic standard of comparison*.

When the task is completely divisible, one of the most widely studied rules is the so-called *equal-distance rule*. It proposes to treat all agents as equally as possible with respect to their *losses*, subject to efficiency. We provide arguments to defend the *up methods* as the discrete version of the *equal distance rule*.

The rest of the paper is structured as follows: In Section 2 we set up the model. In Section 3 we define an allotment procedure: *up methods*. In Section 4 we analyze the properties *up methods* satisfy in and we present our main result. In Section 5 we establish the connections between *up methods* and the *equal-distance rule*.

2 Statement of the model

A preference relation, R, defined over $\mathbb{Z}_+$ is **single-peaked** if there exists an integer number $p(R) \in \mathbb{Z}_+$ (called the **peak** of R) such that, for each $a, b \in \mathbb{Z}_+$,

$$aPb \Leftrightarrow [(b < a < p(R)) \text{ or } (p(R) < a < b)],$$

where P is the strict preference relation induced by R. Let $\mathbb{S}$ denote the class of all single-peaked preferences defined over $\mathbb{Z}_+$. Let $\mathbb{N}$ be the set of all potential agents and $\mathcal{N}$ be the family of all finite non-empty subsets of $\mathbb{N}$.

An allocation problem with single-peaked preferences, or simply a **problem**, is a triple $\boldsymbol{e = (N, T, R)}$ in which a fixed number of units T (called **task**) has to be distributed among a group of **agents**, $N \in \mathcal{N}$, whose **preferences** over consumption are **single-peaked**, $R = (R_i)_{i \in N} \in \mathbb{S}^N$. Let $\mathbb{A}^N$ denote the class of problems with fixed-agent set N, and $\mathbb{A}$ the class of all problems, that is,

$$\mathbb{A}^N = \left\{ e = (N, T, R) \in \{N\} \times \mathbb{Z}_+ \times \mathbb{S}^N \right\}$$

and

$$\mathbb{A} = \bigcup_{N \in \mathcal{N}} \mathbb{A}^N.$$

For each problem, we face the question of finding a division of the task among the agents. An **allocation** for $e \in \mathbb{A}$ is a list of integer numbers, $\boldsymbol{x} \in \mathbb{Z}_+^N$, satisfying the condition of being a complete distribution of the task, i.e., $\sum_{i \in N} x_i = T$. Let $\boldsymbol{X(e)}$ be the set of all allocations for $e \in \mathbb{A}$. A **rule** is a function, $\boldsymbol{F} : \mathbb{A} \longrightarrow \mathbb{Z}_+^N$, that selects, for each problem $e \in \mathbb{A}$, a unique allocation $F(e) \in X(e)$.

3 Up methods

We propose now a very natural rule to solve allocation problems with indivisibilities when preferences are single-peaked. It is a sequential procedure by distributing the task one by one. At each step of the process we need to decide the agent getting the unit. To do that we use a sort of order, called *monotonic standard of comparison*.

A *monotonic standard of comparison* is a linear order (complete, antisymmetric and transitive binary relation) over the cartesian product potential agent-integer number, $\mathbb{N} \times \mathbb{Z}$.[1]

Monotonic standard of comparison $\boldsymbol{\sigma} : \mathbb{N} \times \mathbb{Z} \longrightarrow \mathbb{Z}$ such that: (1) for each $i \in \mathbb{N}$ and each $a \in \mathbb{Z}$, $\sigma(i, a+1) < \sigma(i, a)$ and (2) for each $a, b \in \mathbb{Z}$, if $a > b$, then $\sigma(i, a) < \sigma(j, b)$.[2] Let Σ^M denote the class of monotonic standards of comparison.

In other words, monotonic standards of comparison always give priority to agents with larger integer numbers.

Let $\{(i, a_i)\}_{i \in M}$ be a collection of pairs agent-number. Let $\sigma \in \Sigma$ be a standard of comparison. The **pair with the highest priority** in $\{(i, a_i)\}_{i \in M}$, according to σ, is the pair (i, a_i) such that $\sigma(i, a_i) < \sigma(j, a_j)$ for all $j \in M \setminus \{i\}$.

For each monotonic standard of comparison we construct a rule. As we mentioned this rule is a sequential process. At each step, we allocate one unit of the task, and we decide the agent gaining such a unit by using the monotonic standards of comparison. We call those rules *up methods*.

Up method associated to $\boldsymbol{\sigma}$, $\boldsymbol{U^{\sigma}}$: Let $e \in \mathbb{A}$. Associate to each agent her peak, that is, $a_i = p(R_i)$. Identify the pair with the highest priority according to σ. Give one unit of the task to this agent, and reduce her number by one unit. Identify again the pair with the highest priority according to σ, and proceed in the same way until the task is exhausted.

Next example illustrates how up methods work.

Example 3.1. Assume that the standard of comparison σ is such that, restricted to agents in $N = \{1, 2, 3\}$, it happens that $\sigma(2, x) < \sigma(1, y) < \sigma(3, z)$, for all $x, y, z \in \mathbb{Z}$. Furthermore, $\sigma(1, x) < \sigma(2, x) < \sigma(3, x)$ if x is odd, and $\sigma(1, x) < \sigma(3, x) < \sigma(2, x)$ if x is even. Now, consider the allocation problem where $N = \{1, 2, 3\}$, $T = 8$, and $R = (R_1, R_2, R_3)$ such that $R_2 = R_3$

[1] The notion of standard of comparison was formulated by Young (1994). Here we concentrate in the subclass introduced by Herrero and Martínez (2006a).

[2] If $\sigma(i, a) < \sigma(j, b)$ we will understand that the pair (i, a) has priority over the pair (j, b).

and $p(R) = (1, 3, 3)$. We associate to each agent her peak, that is, we consider the pairs $(1, 1)$, $(2, 3)$, and $(3, 3)$. According to σ, the pair with the highest priority is $(2, 3)$. Then we give one unit to agent 2, and we now consider the new pairs $(1, 1)$, $(2, 2)$, and $(3, 3)$. According to σ, the pair with the highest priority is $(3, 3)$. Then we give one unit to agent 3, we consider the new pairs $(1, 1)$, $(2, 2)$, and $(3, 2)$. By repeating this process until allocating the 8 units, we conclude that $U^{\sigma}(e) = (2, 3, 3)$.

4 Properties

An allocation is *efficient* if there is no other allocation in which all the agents are better off. *Efficiency* requires the rule to select efficient allocations.[3]

Efficiency: For each $e \in \mathbb{A}$, there is no allocation $x \in X(e)$ such that, for each $i \in N$, $x_i R_i F_i(e)$, and for some $j \in N$, $x_j P_j F_j(e)$.

The principle of efficiency is equivalent to asking for each agent to consume, no more than her peak when the task is too little, and no less than her peak when the task is too much. That is, if F is efficient, $F_i(e) \leq p(R_i)$ for each $i \in N$ when $\sum_{i \in N} p(R_i) \geq T$, and $F_i(e) \geq p(R_i)$ for each $i \in N$ when $\sum_{i \in N} p(R_i) \leq T$.

In any rationing framework, a minimal fairness condition is always desirable. The first try would be the *equal treatment of equals* principle. It says that agents with identical preferences should be indifferent among their respective allocations. Paired with the requirement of efficiency, it simply means that agents with identical preferences receive the same amount. It is easy to check that no rule can fulfill equal treatment of equals in the context of problems with indivisibilities. Young (1994), and Herrero and Martínez (2006b) formulate a milder version of this condition: *balancedness*. It postulates that equal agents should be treated, if not equal, at least as equal as possible. Balancedness requires that the awards of equal agents to differ, at most, by one unit (representing this unit the size of the indivisibility).

Balancedness: For each $e \in \mathbb{A}$ and each $\{i, j\} \subseteq N$, if $R_i = R_j$ then $|F_i(e) - F_j(e)| \leq 1$.

It happens many times that the number of agents is so large that collecting all the information regarding to the whole preferences becomes an issue. *Peaks only* is a condition of informational efficiency. It requires agent's allocation to depend only on her peak.

Peaks only: For each $e = (N, T, (R_i, R_{-i})) \in \mathbb{A}$ and each $e' = (N, T, (R'_i, R_{-i})) \in \mathbb{A}$ such that $p(R'_i) = p(R_i)$, then $F_i(e) = F_i(e')$.

The next principle, *ar-truncation*, can be interpreted as an instance of a general principle of independence of irrelevant alternatives. Given $e \in \mathbb{A}$, let $ar(e) = \frac{\sum_{j \in N} p(R_j) - T}{n}$. The number $ar(e)$ is simply the average rationing of the task among the agents in N. *Ar-truncation* requires any information on the agents' preferences below $ar(e)$ to be ignored. In consequence, all those

[3]It is worth noting that if Condition (2) in the definition of monotonic standard of comparison is not satisfied, the up method may violate efficiency.

problems whose preferences coincide in $[ar(e), +\infty[$ are indistinguishable.

Ar-truncation: For each $e = (N, T, R) \in \mathbb{A}$ and each $e' = (N, T, R') \in \mathbb{A}$, if for each $i \in N$, $R_i = R'_i$ on $[ar(e), +\infty[$, then, $F(e) = F(e')$.

Imagine now that when estimating the value of the task this falls short, so that the real value is larger than expected. Then two possibilities are open, either to forget about the initial allocation and just solve the new problem, or keep the tentative allocation and then allocate the rest of the task among the agents, after adjusting the preferences by shifting them by the amount already obtained. *Agenda independence* requires the final allocation not to depend on this timing.

Agenda independence For $e = (N, T, R) \in \mathbb{A}$ and each $T' \in \mathbb{Z}_{++}$, $F(e) = F(N, T', R) + F(N, T - T', R')$, where $R'_i = \pi^{F_i(N,T',R)}(R_i)$.[4]

Next property refers to an stability condition with respect to changes in population. Suppose that, after solving the problem $e = (N, T, R) \in \mathbb{A}$, a proper subset of agents, $S \subset N$, decides to reallocate the total amount they have received, that is, they face a new allocation problem: $(S, \sum_{i\in S} a_i, R_S)$, where $R_S = (R_i)_{i\in S}$ and a is the allocation corresponding to apply the rule to the problem e. *Consistency* requires each agent $i \in S$ to receive the same amount of units in problem $(S, \sum_{i\in S} a_i, R_S)$ as she did in problem e. In other words, the new reallocation is only a restriction to the subset S of the initial one.[5]

Consistency: For each $e \in \mathbb{A}$, each $S \subset N$, and each $i \in S$, $F_i(e) = F_i(S, \sum_{j\in S} F_j(e), R_S)$.

The family of up methods satisfy all the aforementioned properties. Moreover, in our main result, we obtain that such a family is the unique one satisfying those properties. The proof, preceded by some technical results, is relegated to Appendix B.

Theorem 4.1. *A rule F satisfies balancedness, peaks only, agenda independence, ar-truncation, and consistency if and only if there exists a monotonic standard of comparison $\sigma \in \Sigma^M$ such that $F = U^\sigma$.*

5 Relations between the discrete and the continuum

The properties in Theorem 4.1 characterizing the family of up methods are in line with the characterization of the equal-distance rule in Herrero and Villar (1999), with identical proviso.[6] This fact suggests a relationship between our methods and the equal-distance rule. Actually, any up method can be interpreted as a discrete version of the equal-distance rule. This statement is

[4]For a given $a \in \mathbb{Z}$, $\pi^a : \mathbb{S} \longrightarrow \mathbb{S}$ is defined as follows: For each $R \in \mathbb{S}$, $x\pi^a(R)y$ iff $(x + a)R(y + a)$. Given $R \in \mathbb{S}$, we call $\pi^a(R)$ the *shifting* of R by a.

[5]The reader is referred to Thomson (2004) for a widely exposition of consistency and its converse.

[6]Under the assumption that the task were completely divisible, one of the most widely studied rules is the so-called equal distance rule. The underlying idea of this rule is equality on losses, above or below, depending on the case, with respect to the peaks.

Equal distance rule, ***ed***: For each $e \in \mathbb{A}$, selects the unique vector $ed(e) \in \mathbb{R}^N$ such that $ed(e) = \max\{0, p(R_i) + \lambda\}$ for some $\lambda \in \mathbb{R}$.

supported in the following. For any problem, the allocation prescribed by the equal-distance rule is the ex-ante expectations of the agents under the application of up methods, if all monotonic standard of comparison are equally likely.

Proposition 5.1. *Let $e \in \mathbb{A}$. Let Σ_e^M denote the subset of Σ^M of the different partial standards involved in problem e.*[7] *Then*

$$\frac{1}{|\Sigma_e^M|} \sum_{\sigma \in \Sigma_e^M} U^{\sigma}(e) = ed(e).$$

[7] In Σ^M we consider all possible standards over $\mathbb{N} \times \mathbb{Z}_{++}$. Notice that, for a given e, not all of them rank the pairs (i, a_i) involved in that particular problem in different ways. Σ_e^M denotes precisely the subset of those different standards.

Appendix A. On the tightness of characterization result

We present now a collection of examples to illustrate the independence of properties used in Theorem 4.1.

Example 5.1 (Peaks Only)**.** Let us define the rule F^{α} for the two-agent case, $N = \{i, j\}$. Let us define the order $\alpha : N \times \mathbb{Z} \times \mathbb{S}^2 \longrightarrow \mathbb{Z}_{++}$ such that, if $x > y$, then $\alpha(\cdot, x, \cdot) < \alpha(\cdot, x, \cdot)$. And, if $x = y$, then

$$R_i = R_j \Rightarrow \alpha(i, x, (R_i, R_j)) < \delta(j, x, (R_i, R_j))$$
$$R_i \neq R_j \Rightarrow \alpha(j, x, (R_i, R_j)) < \delta(i, x, (R_i, R_j))$$

The order α determines, in case of having only one unit, the agent who gets it. It will depend on the agent, the peaks, and the preferences. To obtain the allocation prescribed by the rule associated to that order α, F^{α}, we proceed in the following way. Let us consider the problem $(\{i, j\}, T, (R_i, R_j))$. Then, identify the agent with the smallest α for the problem, let us say agent i. Give one unit of the task to i. Shift agent i's preferences by a unit to $R'_i = \pi^{F_i(\{i,j\},1,(R_i,R_j))}(R_i)$. In the new problem, $(\{i, j\}, T - 1, (R'_i, R_j))$, proceed in the same way. Repeat this process until the task runs out.

Example 5.2 (Balancedness)**.** Select one particular agent $i \in \mathbb{N}$ from the set of all potencial agents. For each $\sigma \in \Sigma^M$, the rule H^{σ} is defined as

$$H_j^{\sigma}(e) = \begin{cases} U_j^{\sigma}(e) & \text{if } \sum_{k \in N} p(R_k) \geq T \\ p(R_i) & \text{if } \sum_{k \in N} p(R_k) < T \text{ and } j = i \\ U_j^{\sigma}(N \setminus \{i\}, T - p(R_i), R_{N \setminus \{i\}}) & \text{if } \sum_{k \in N} p(R_k) < T \text{ and } j \neq i \end{cases}$$

Example 5.3 (Agenda independence)**.** Let $\sigma \in \Sigma^M$, then

$$E^{\sigma}(e) = \begin{cases} U^{\sigma}(e) & \text{if } \sum_{j \in N} p(R_j) \leq T \\ TS^{\sigma}(e) & \text{if } \sum_{j \in N} p(R_j) \geq T \end{cases}$$

where TS^{σ} is the M-temporary satisfaction method associated to σ define by Herrero and Martínez (2006b).

Example 5.4 (ar-truncation)**.** Let $\succ: \mathbb{N} \longrightarrow \mathbb{Z}_{++}$ be an order defined over the set of potential agents such that agent labeled i has priority over agent labeled $i + 1$, i.e., $i \succ i + 1$. And let $\sigma \in \Sigma^M$ a monotonic standard of comparison. Both $\succ$ and σ are independent. Now, for each problem $e \in \mathbb{A}$, the rule K works as follows. If no subset of agents have equal peaks (i.e., all the peaks are different), then we give one unit of the task according to the up method associated to σ, U^{σ}, and we reduce one unit the peak of the agent who has received the unit. If a subset of agents, let say S, have equal peaks, then we give one unit of the task to the agent in S who has the smallest label (that is, the agent in S with the highest priority according to $\succ$) among all of them involved in S. If there were two or more subsets of agents, let us say S and T, with equal peaks, then we give unit of the task to the agent with the smallest label among all of them involved in S and T. After that, we reduce this agent's peak by one unit. We repeat the process until the task runs out.

Example 5.5 (Consistency)**.** This rule, F, can be defined as follows. Let $\sigma_1, \sigma_2 \in \Sigma^M$ be two different monotonic standards such that $\sigma_1(i,x) < \sigma_1(i+1,x)$ and $\sigma_2(i+1,x) < \sigma_2(i,x)$. Then, we define the solution $F^{(\sigma_1,\sigma_2)}$ as

$$F^{(\sigma_1,\sigma_2)}(e) = \begin{cases} U^{\sigma_1}(e) & \text{if } |N| = 2 \\ U^{\sigma_2}(e) & \text{otherwise} \end{cases}$$

Appendix B. Proofs of the results

Lemma 5.1 (Elevator Lemma, (Thomson (2004))). *If a rule F is consistent and coincides with a conversely consistent rule F' in the two agent case, then it coincides with F' in general.*[8]

Lemma 5.2 (Herrero and Martínez (2006a)). *Efficiency, one-sided resource monotonicity, and consistency together imply converse consistency.*[9]

Proof of Theorem 4.1.

It is easy to check that each up method satisfies the properties. Conversely, let F be a rule satisfying the five properties.

Step 1. *Definition of the standard of comparison.* Let us define the order $\sigma \in \Sigma^M$ as follows

$$a > b \Rightarrow \sigma(i, a) < \sigma(j, b)$$
$$a = b \Rightarrow [\sigma(i, a) < \sigma(j, b) \Leftrightarrow F_i(\{i, j\}, 1, (R_i, R_j)) = 1],$$

where R_i and R_j are two single-peaked preference relations such that $p(R_i) = a = b = p(R_j)$ (by *peaks only* it is enough to consider the peaks). It is straightforward to see that such a σ is complete and antisymmetric. Let us show that σ is transitive. Suppose that there exist $\{i, j, k\} \subseteq \mathbb{N}$ such that $\sigma(i, x) < \sigma(j, y)$, $\sigma(j, y) < \sigma(k, z)$, but $\sigma(i, x) > \sigma(k, z)$. By construction and *peaks only*, this can only happen when $x = y = z$. By the definition of σ, in such a case, $F_i(\{i, j\}, 1, (R_i, R_j)) = 1$, $F_j(\{j, k\}, 1, (R_j, R_k)) = 1$, and $F_k(\{k, i\}, 1, (R_k, R_i)) = 1$, where $p(R_i) = p(R_j) = p(R_k) = x = y = z$. Consider the problem $(\{i, j, k\}, 2, (R_i, R_j, R_k))$. There are only three possible allocations: $(1, 1, 0)$, $(1, 0, 1)$, and $(0, 1, 1)$. Suppose that $F(\{i, j, k\}, 2, (R_i, R_j, R_k)) = (1, 1, 0)$, by *consistency*, $F_k(\{i, k\}, 1, (R_i, R_k)) = 0$, achieving in this way a contradiction with $F_k(\{i, k\}, 1, (R_i, R_j)) = 1$. An analogous argument is applied if $F(\{i, j, k\}, 2, (R_i, R_j, R_k)) = (1, 0, 1)$, or if $F(\{i, j, k\}, 2, (R_i, R_j, R_k)) = (0, 1, 1)$. Therefore $\sigma(i, x) < \sigma(k, z)$, and then σ is transitive.

[8]Let us consider an allocation for a problem with the following feature: For each two-agents subset, the rule chooses the restriction of that allocation for the associated reduced problem to this agent subset. *Converse consistency* requires the allocation to be the one selected by the rule for the original problem. This property was formulated by Chun (1999) in the context of claims problems.

Let $c.con(e; F) \equiv \{x \in \mathbb{Z}^N_+ : \sum_{i \in N} x_i = T$ and for all $S \subset N$ such that $|S| = 2, x_S = F(S, \sum_{i \in S} x_i, R_S)\}$

Converse consistency: For each $e \in \mathbb{A}$, $c.con(e; F) \neq \phi$, and if $x \in c.con(e; F)$, then $x = F(e)$.

It is worth noting that converse consistency implies consistency. Besides, Thomson (2004) formulated the following useful result involving both properties.

[9]*One-sided resource monotonicity*, considers the case in which the change in the task does not alter the type of rationing associated to the initial problem, i.e, if initially we have to ration labor, it is still labor to be rationed after the task increasing, or else, if in the initial problem we have to ration leisure, then again, we have too much labor to allocate even after the decreasing of the task. In either case, the property states that no agent should suffer.

One-sided resource monotonicity: For each $e, e' \in \mathbb{A}$ such that $e = (N, T, R)$ and $e' = (N, T', R)$. If (a) $\sum_{j \in N} p(R_j) \geq T' > T$, or (b)$\sum_{j \in N} p(R_j) \leq T' < T$. Then for each $i \in N$, $F_i(e') R_i F_i(e)$.

Step 2. *Let us prove now that* $F = U^\sigma$. It is straightforward that U^σ is efficient, one-sided resource monotonic, and consistent. Then, by Lemma 5.2, U^σ is conversely consistent. Therefore, in application of Lemma 5.1, it is sufficient to show that both F and U^σ coincide in the two-agent case. Then, let us consider the problem $e = (S, T, R) \in \mathbb{A}$ where $S = \{i, j\}$. Without loss of generality we can assume that $p_i \equiv p(R_i) \le p(R_j) \equiv p_j$. Suppose first that $p_i = p_j$. By *peaks only, balancedness, agenda independence*, and the definition of the standard, $F(e) = U^\sigma(e)$. Let us suppose now that $p_i \neq p_j$. We distinguish now the following cases:

Case 1. If $p_i + p_j = T$. Let us show that $F(S, T, (R_i, R_j)) = (p_i, p_j) = U^\sigma(S, T, (R_i, R_j))$. By *ar-truncation*, $F(S, p_j - p_i, (R_i, R_j)) = (0, p_j - p_i)$. Once we have allotted the amount $p_j - p_i$, both agents have the same preference relation: $R'_i = R'_j$, and $T - (p_j - p_i) = 2p_i$ units remain to allocate. By *balancedness*, $F(S, 2p_i, (R'_i, R'_j)) = (p_i, p_i)$. In application of *agenda independence*, $F(S, T, (R_i, R_j)) = F(S, p_j - p_i, (R_i, R_j)) + F(S, 2p_i, (R'_i, R'_j)) = (0, p_j - p_i) + (p_i, p_i) = (p_i, p_j)$.

Case 2. If $p_i + p_j < T$. Let us define $T' = p_i + p_j$. Then $F(S, T', R) = (p_i, p_j) = U^\sigma(S, T', R)$ by Case 1. Once we have allotted the amount T', both agents have the same preference relation: $R'_i = R'_j$. And then $F(S, T - T', (R'_i, R'_j)) = U^\sigma(S, T - T', (R'_i, R'_j))$. By *agenda independence*, $F(e) = F(S, T', R) + F(S, T - T', (R'_i, R'_j)) = U^\sigma(S, T', R) + U^\sigma(S, T - T', (R'_i, R'_j)) = U^\sigma(e)$.

Case 3. If $p_i + p_j > T$. If T is such that $0 \le T \le p_j - p_i$, then $ar(e) \le p_i$. By *ar-truncation*, $F(e) = (0, T) = U^\sigma(e)$. If T is such that $p_j - p_i \le T \le p_i + p_j$, then, by *agenda independence*, $F(e) = F(S, p_j - p_i, R) + F(S, T - (p_j - p_i), R')$, where $R'_i = R'_j$. Note that, by *ar-truncation*, $F(S, p_j - p_i, R) = (0, p_j - p_i) = U^\sigma(S, p_j - p_i, R)$. By *balancedness* and the definition of the standard, $F(e) = F(S, p_j - p_i, R) + F(S, T - (p_j - p_i), R') = U^\sigma(S, p_j - p_i, R) + U^\sigma(S, T - (p_j - p_i), R') = U^\sigma(e)$.

Then, F coincides with U^σ in the two agents case, and therefore they do so in general.

Proof of Proposition 5.1.

On one hand, it is known that the equal-distance rule satisfies *converse consistency*. On the other hand, it is easy to check that the up methods are *consistent*. Then the average given by the left hand side in the formula is also consistent (see Thomson (2004)). By using Lemma 5.1 it is enough to consider the two-agent case. But it is straightforward that in this case both the equal-distance rule and the average coincide.

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