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Optimization in non-standard problems. An application to the provision of public inputs

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Optimization in non-standard problems. An application to the provision of public inputs*

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Abstract

This paper describes a new method for solving non-standard constrained optimization problems for which standard methodologies do not work properly. Our method (the Rational Iterative Multisection -RIM- algorithm) consists of different stages that can be interpreted as different requirements of precision by obtaining the optimal solution. We have performed an application of RIM method to the case of public inputs provision. We prove that the RIM approach and comparable standard methodologies achieve the same results with regular optimization problems while the RIM algorithm takes advantage over them when facing non-standard optimization problems.

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1 Introduction

The use of numerical methods is a standard feature in many areas of Economics. This is due, among other things, to the need of finding clear-cut results beyond the ambiguity of algebraically-developed analyses and the interest for replicating the characteristics of real world in policy-oriented exercises. A good example of this is given by the debate on the optimal level of public spending when distortionary taxation is involved. On the basis of the paper by Atkinson and Stern (1974), contributions such as Gaube (2000, 2005), Chang (2000) or Gronberg and Liu (2001) discuss when the first-best level of public goods exceeds the second-best level. The point here is that it is not straightforward to derive a general rule to elucidate analytically the question and numerical exercises have to be carried out.

A similar circumstance is found in the presence of public inputs: an analytic discussion concerning the optimal levels of public inputs on the basis of optimality rules is not conclusive and numerical simulations are required. Moreover, the case of productivity-enhancing public spending shows particular features and deserves a specific treatment (Feehan and Matsumoto, 2000, 2002; Martinez and Sanchez, 2008). Indeed, under plausible assumptions related to the way through which the public input enters the production function, standard numerical methods may fail out. Particularly, when increasing returns to scale in all the production factors are present (then the public input is named factor-augmenting), non-convexities arise and standard methods such as Newton-Raphson or Nelder-Mead algorithms may have problems to achieve the solution.

This paper introduces a new method, the Rational Iterative Multisection (RIM, hereafter) algorithm, for solving non-standard constrained optimization problems, overcoming the caveats of conventional algorithms. This new approach is applied to the controversy on the optimal level of public inputs. The paper gives some insights on this issue under different tax settings.

Our method is based on evaluations of the objective function in a multisection of the initial set of possible values, reaching the optimum through an iterative process. Therefore, our proposal is related to direct search methods in which the aim is to find global solutions by comparing the values of the objective function at different points (see Casado et al. (2000), Kolda (2003), Burmen et al (2005), Mathews and Fink (2004), Coope and Price (2000), and others). With the procedure we propose here, the level of precision with which the constraints of the problem are fulfilled becomes a crucial criterion. Starting from an initial set of decision variables, the RIM method selects the compatible values where the constraints are fulfilled with a certain precision. Subsequent evaluations of the objective function lead to choose the optimal values of decision variables for each level of precision. Moreover, non-optimal solutions (and information to assess how far they are from the local or global optima) are obtained. In this sense, the path followed by the iterative process towards the global optimum is clearly shown and a wide-ranging set of non-optimal values according to the precision required is provided, making richer the discussion of results. Finally, if multiple equilibria are detected, they are discriminated by using this additional information.

With the aim of exploring its consistency, we compare this new method to a standard methodology which has been widely used in general equilibrium models, the Newton-Raphson method (NR, hereafter), in a context where its regularity requirements are satisfied. In this framework, our method achieves the same results at optimum than NR. The point is that RIM algorithm is unaffected by situations in which NR method does not properly work. Many of these situations are linked to non-convex problems, such as increasing returns to scale in the production function, which is the case we study as application in this paper.

Additionally, we have adapted the RIM algorithm to unconstrained optimization problems by using a different selection criterion in each stage which detect the changes in monotonicity of the objective function. Moreover, we check the performance of RIM approach within this framework by comparing to the standard methodology of Nelder-Mead algorithm (NM, hereafter), and we obtain the same results. This finding shows the flexibility of our proposal.

The structure of the paper is as follows. Section 2 explains how the RIM methodology works with a brief description of the problem to be solved and other methods used as reference (NR and NM). Section 3 presents an application of RIM algorithm in which both regular and non-regular constrained optimization problems are solved; a discussion of the results is also included. Finally, section 4 concludes.

2 General description of the methods

In this section we set up the general framework of the problem to be solved and the three methods used in its resolution as well. Obviously, we focus our attention upon the RIM algorithm given that the others two procedures are well-known.

2.1 The problem

The problem we are interested in solving is:

$$\begin{cases} \max f(u,p) \\ s.t.: R(u,p) = 0 \\ u \in U,\ p \in P. \end{cases} \tag{1}$$

where $f : U \times P \subseteq \mathbb{R}^n \times \mathbb{R}^z \longrightarrow \mathbb{R}$ is the objective function to be optimized, $R : U \times P \subseteq \mathbb{R}^n \times \mathbb{R}^z \longrightarrow \mathbb{R}^m$ the set of constraints of the problem, which are assumed to be differentiable[1], U the set of feasible values for the decision variables (u) (which can be an interval or the union of several intervals), and P the set of parameters fixed throughout all the process (p). The number of decision variables is denoted by n, z is the number of parameters and m the number of constraints.

2.2 Newton-Raphson (NR) method

This iterative method has at least two advantages: its high convergence speed and its simple structure. Using the properties of the gradient, it is straightforward to achieve the point in which the objective function is maximized. The performance of NR method is simple:

Let $\pounds$ be the function to be optimized[2]:

$$\begin{array}{rcl} \pounds : X \subseteq \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \\ x & \longrightarrow & \pounds(x), \end{array}$$

where $\pounds$ is differentiable. The method is used to solve $\pounds(x) = 0$. Given $x_0 \in X \subseteq \mathbb{R}^n / \exists \nabla \pounds(x_0)^{-1}$, the iterative process has the following steps for each $i > 0$:

1. Evaluate $\nabla \pounds(x_i)$.

2. If $\exists \nabla \pounds(x_i)^{-1}$, the point to be used in the next iteration is computed[3]: $x_{i+1} = x_i - \pounds(x_i) * \nabla \pounds(x_i)^{-1}$.

3. The stop criterion is defined as follows: given $\epsilon > 0$, if $\|x_{i+1} - x_i\| < \epsilon$, then x_{i+1} is the root of the function; otherwise, the procedure continues until this condition holds[4].

[1] This assumption is required for using gradient-based algorithms. RIM method works properly even if this property is not satisfied.

[2] The problem (1) has been adapted to this nomenclature using the lagrangian function.

[3] If $n \neq m$, the generalized inverse is considered.

[4] There are others possibilities for setting the stopping criterion. For example, ϵ could be defined as $\|\frac{x_{i+1} - x_i}{x_{i+1}}\| < \epsilon$.

Further information on NR method can be found in the large bibliography existing about this approach (see Beninga (1989), Kelley (2003) and others). However, this widely-used method has some relevant caveats. First, it does not work properly when the domain of objective function is non-convex. Second, it is necessary to have the gradient of function $\pounds$ different to zero; otherwise, the method does not converge. Third, if the objective function has multiple solutions, there exists the risk of jumping from a root near the initial point to other possible solutions, neglecting closer and more accurate solutions. And finally, convergence problems may appear when several local optima are involved in the problem.

2.3 Nelder-Mead (NM) algorithm

This algorithm is probably the most popular direct search method. According to Burmen et al (2005), the performance of this approach could be shortly summarized as follows.

Let g the function to be minimized[5]:

$$\begin{array}{ccc} g : X \subset \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ x & \longrightarrow & g(x), \end{array}$$

The algorithm manipulates a set of $n+1$ vertices in $\mathbb{R}^n$ (a simplex[6]) which are ordered according to the objective function values so that $g(x^1) \leq \cdots \leq g(x^{n+1})$. The centroid of the n vertices with the lowest values is defined as $x^{cb} = \sum_{i=1}^{n} x^i$. The centroid and x^{n+1} define the line along which candidates are examined for replacing the vertex with the highest objective function value. The examined points can be expressed as $x(\gamma) = x^{cb} + \gamma\left(x^{cb} - x^{n+1}\right)$. They are denoted by x^r, x^e, x^{oc}, and x^{ic} with the corresponding values of γ denoted by γ^r, γ^e, γ^{oc}, and γ^{ic}. They are often referred to as the reflection, expansion, outer contraction, and inner contraction points. Under certain circumstances, the simplex is shrunk towards x^1 using the formula $x^1 + \gamma^S\left(x^{cb} - x^{n+1}\right)$ for $i = 2, 3, \ldots, n+1$. Above values of γ satisfy these requirements

$$0 < \gamma^r < \gamma^e, \qquad \gamma^e > 1, \qquad 0 < \gamma^{oc} < 1, \qquad -1 < \gamma^{ic} < 0, \qquad 0 < \gamma^S < 1$$

Nelder and Mead proposed originally the following values: $\gamma^r = 1$, $\gamma^e = 2$, and $\gamma^{oc} = -\gamma^{ic} = \gamma^S = 0.5$. The typical iteration of this algorithm can be described as follows:

[5] The problem (1) has been adapted to this nomenclature using $g = -f$

[6] A triangule when $n = 2$.

1. Order the simplex.

2. Evaluate $g^r = g(x^r)$. If $g^r < g^1$, then evaluate $g^e = g(x^e)$. Therefore, x^{n+1} is replaced by x^e if $g^e < g^r$; otherwise, x^{n+1} is replaced by x^r.

3. If $g^1 < g^r < g^n$, then x^{n+1} is replaced by x^r.

4. If $g^n \leq g^r < g^{n+1}$, then evaluate $g^{oc} = g(x^{oc})$. Therefore, x^{n+1} is replaced by x^{oc} if $g^{oc} \leq g^{n+1}$.

5. If $g^{n+1} \leq g^r$, then evaluate $g^{ic} = g(x^{ic})$. Therefore, x^{n+1} is replaced by x^{ic} if $g^{ic} \leq g^{n+1}$.

6. If x^{n+1} is not replaced, shrink the simplex towards x^1.

Further details concerning this method can be found in the large bibliography existing about it (see, for instance, Kelley (1999), Coope and Price (2000) and others). Convergence problems have been detected even for smooth functions of low dimension. McKinnon (1998) presents a family of functions where this method does not work properly because the simplices become arbitrarily narrow.

2.4 Rational Iterative Multisection method

The new method we propose belongs to the family of direct search methods (see Casado et al. (2000), Kolda (2003), Kelley (1999), Mathews and Fink (2004), among others), and consists of an iterative subdivision of the initial decision set of variables. Previously, we have selected the points of the grid that satisfy the constraints with a certain precision in each stage. This process continues until the maximum previously-fixed precision is achieved. Whereas most of the direct search methods do not consider different levels of precision (because they usually solve unconstrained optimization problems) our numerical approach has been adapted to take into account the precision with which the constraints are held. Hence, the constraints of the problem become specially relevant when RIM approach is applied. A formal description of RIM algorithm is next.

The definition of some instrumental but essential parameters is necessary as long as the resolution of the problem consists of using different stages in which a certain precision and a bandwidth for each interval are set.

Definition 1 *Let S be the number of stages, then:*

- Precision path $E = [E_1, ..., E_S]$ *is the vector containing the precision required in the different stages of resolution.*

- Bandwidth path $B = [B_1, ..., B_S]$ *is the vector formed by the length of the space considered between two points of the grid in different stages of resolution.*

Both variables are interrelated because B_s refers to the bandwidth used for achieving the precision E_s in U_s, *i. e.*, the feasible values set for the stage s. The choice of these parameters is crucial and may affect the performance of the method.

Definition 2 *Given the problem (1), $\epsilon > 0$ and the set $W \subseteq U$, let $C(\epsilon, W)$ be the set of compatible values in which the constraints are fulfilled with the precision ϵ, i.e.,*

$$C(\epsilon, W) = \{w \in W \mid \|R(w, p)\|_{\max} < \epsilon\}$$

Translating the problem (1) to the new nomenclature, the problem to be solved is:

$$\begin{cases} \max f(c, p) \\ c \in C(\epsilon, U), p \in P \end{cases} \tag{2}$$

Definition 3 *Let $\tilde{c} \in C(E_S, U_S)$ be the solution to the problem (2), that is, the value which satisfies the condition:*

$$f(\tilde{c}, p) \geqslant f(c, p), \forall c \in C(E_S, U_S) \tag{3}$$

The implementation of this general procedure must take account several considerations related to efficiency in computation. And here, the number of decision variables is the key issue. The following nomenclature is now introduced for the sake of simplicity: RIMn will refer to the RIM method which considers n decision variables. For instance, RIM2 refers to the method facing a problem with two decision variables. In order to make easier the understanding of the general procedure, the RIM2 is considered next, and $c = (c^1, c^2)$.[7] Let us consider I as an arbitrary interval of U_s, *i.e.*, the set of feasible values for the stage s.[8] The resolution for the remaining intervals is analogous to this one. Applying this particular notation to the problem (2) yields the following:

[7]RIM1 method is obviously simpler than RIM2 method, but it does not allow to explain the particular stages in which we are interested.

[8]$I \subseteq U_s \subseteq \mathbb{R}^2$. The method will converge better or worse towards the optimal values depending on the initial set of values. Therefore, it is useful to select the initial range of values with some rationality.

$$\begin{cases} \max f(c,p) \\ c \in C(E_s, I), p \in P \end{cases} \tag{4}$$

With the aim of transforming the continuous problem (4) into a discrete problem, the interval I is subdivided according to the parameter $B_s = (B_s^1, B_s^2)$. Hence, the vectors $C^1 = \{c_i^1\}$ and $C^2 = \{c_j^2\}$ are built using the above information:

$$c_i^1 = \underline{\mathrm{I}}_1 + (i-1)B_s^1, i = 1, ..., D_s^1 + 1$$
$$c_j^2 = \underline{\mathrm{I}}_2 + (j-1)B_s^2, j = 1, ..., D_s^2 + 1,$$

where $\bar{I}_k = \max\{c^k | c \in I\}$, $\underline{\mathrm{I}}_k = \min\{c^k | c \in I\}$, and $D_s^k = \dfrac{\bar{I}_k - \underline{\mathrm{I}}_k}{B_s^k}$ are the number of subdivisions, $k = 1, 2$. Depending on the problem, it may be useful to set $B_s^2 = B_s^1$ to obtain the same scale in the different decision variables[9].

With these vectors, the grid for the interval I_s in this stage is $I_s = C^1 \times C^2$. On the basis of these points all the variables of the problem are evaluated, constraints R included. Thus, for each value of the first decision variable, the values of the other which satisfy the constraints R with a precision E_s are chosen. In other words, for each c_i^1, the set of good values of the other decision variable c_j^2, $G(c_i^1, E_s)$, is defined. Formally,

$$G(c_i^1, E_s) = \{c_j^2 \in C^2 / \|R(c_i^1, c_j^2)\|_{\max} < E_s\},$$

and grouping the different c_i^1's:

$$G_1(E_s) = \{c_i^1 \in C^1 / G(c_i^1, E_s) \neq \emptyset\}.$$

Using this notation, we find out the solution of problem (4) in the stage s solving the next problem:

$$\begin{array}{c} max(f(c_i^1, G(c_i^1, E_s))) \\ s.t. : c_i^1 \in G_1(E_s) \end{array} \tag{5}$$

It requires to evaluate the objective function at the points satisfying the constraints with the required precision E_s. Additionally, this strategy allows to get a ranking of results in the intermediate stages, and shows one of key features of RIM algorithm compared to others numerical methods[10].

[9] For instance, 10 points for $c^1$and 40 for c^2 when $(c^1, c^2) \in [0,1] \times [0,4]$.

[10] The intermediate solutions can be interpreted as solutions of the problem for less strict levels of precision.

Whereas the intra-stage procedure has been brieftly described above, several comments are necessary to provide some details on the inter-stage algorithm (the step from stage s to stage $s+1$). The process must continue searching for values in which the constraints R hold with the required precision E_{s+1} starting from the discrete set $G_1(E_s) \times \bigcup_{i=1}^{D_s^1+1} G(c_i^1, E_s) \subset C(E_s, I_s)$. Thus, for each c_i^1, we form areas around these values in the following way:

- For c_i^1, the RIM algorithm forms the interval: $[c^1_{\max\{i-1,1\}}, c^1_{\min\{i+1,D_s^1+1\}}]$.
- With respect to the second decision variable, the coordinates which belong to $G(c_i^1, E_s)$ are available[11]. Grouping the consecutive numbers obtained, the different areas where RIM2 approach will search in the next stage are obtained considering the minimum $(\underline{h}_q^i)$ and the maximum $(\bar{h}_q^i)$ coordinate of each consecutive subsequence $q = 1, ..., Q_i$, where Q_i is the number of subsequences[12].

Analytically, the process could be summarized as follows: $\forall c_i^1 \in G_1(E_s)$, $\forall q = 1, ..., Q_i$, RIM2 approach chooses $[c^2_{\underline{h}_q^i-1}, c^2_{\bar{h}_q^i+1}]$, *i. e.*,

$$\bigcup_{c_i^1 \in G_1(E_s)} \bigcup_{q=1}^{Q_i} [c^1_{\max\{i-1,1\}}, c^1_{\min\{i+1,D_s^1+1\}}] \times [c^2_{\underline{h}_q^i-1}, c^2_{\bar{h}_q^i+1}] \subset U_{s+1}$$

The union of all the intervals built for each interval of U_s will form U_{s+1}, that is,

$$U_{s+1} = \bigcup_{I \in U_s} \bigcup_{c_i^1 \in G_1(E_s)} \bigcup_{q=1}^{Q_i} [c^1_{\max\{i-1,1\}}, c^1_{\min\{i+1,D_s^1+1\}}] \times [c^2_{\underline{h}_q^i-1}, c^2_{\bar{h}_q^i+1}]$$

Finally, the optimal solution will be achieved among the values of this interval by evaluating the objective function. In summary, the iterative process followed by RIM algorithm sorts efficiently (grouping bordering areas) the subsequent initial sets according to the level of precision the problem requires. For an intuitive explanation of the transition inter-stages procedure of RIM method, see the Appendix.

RIM algorithm has modified our initial problem (1) into a discrete problem. Although RIM method does not invert this process, we can compute

[11] Obviously, this step would not be necessary for RIM1 approach.

[12] Each consecutive subsequence will form an independent area.

how the objective function and other relevant variables are affected when slight changes in the decision variables occur. The concept of elasticity is a useful tool to explore this issue. The elasticity of Y with respect to the variable X (e_X^Y) is defined as follows:

$$e_X^Y \;= (\triangle Y/Y)/(\triangle X/X)$$

Therefore, RIM approach is useful to study the sensitivity of optimal values to changes in decision variables. Using this concept of elasticity, a comparison of the effects caused by desviations from the optimal values could be carried out. Following the example of elasticity, lower absolute values of e_X^Y imply that the solution achieved is more reliable because there exist less incentives to take different options than the optimal values. RIM algorithm takes advantage here over other methods because this sensitivity analysis is done during the resolution process.

The RIM method works properly even for non-regular optimization problems. The following propositon shows that even a weak assumption on the constraints of the problem ensures the suitable performance of our proposal.

Proposition 1 *Let us consider (1). If there exists k subsets of U:* $U_1, ..., U_k$ ($U = \bigcup_{i=1}^{k} U_i, U_i \cap U_j = \emptyset$ *if* $i \neq j$), *such that R is monotonic in* $U_i, \forall i = 1, ..., k.$, *a solution for this problem can be found using the Rational Iterative Multisection algorithm.*

Proof. Initially, consider the k subintervals in which the function R is monotonic. Then, next stages will increase the level of precision to be satisfied depending on the minimum value of the function R. Finally, the solution for problem (1) is achieved.[13] ■

3 The optimal level of public inputs. An application of RIM method

In this section, RIM algorithm is used to solve an optimization problem where non-regular conditions may be present. In particular, we shed some light on a controversial issue, namely, the optimal level of public spending under different tax settings. The debate comes from the literature on theory of public

[13] A solution could be obtained even if $R > 0$, depending on the desired level of precision. For instance, consider $R \;/\; min(R) = 10^{-6}$.

goods (Samuelson, 1954; Pigou, 1947), with extensions until present (Gaube, 2000, 2005; Chang, 2000). The underlying idea is that using distortionary taxation leads to an optimal level of public goods below its first-best level.

In this paper we translate this debate to the case of public inputs. The point is that the way through which the public input enters the production function is crucial for this controversy and its resolution. Particularly, dealing with factor-augmenting public inputs implies to solve a non-convex optimization problem as long as increasing returns to scale appear in the production side of the economy.

Martinez and Sanchez (2008), using the approach by Gronberg and Liu (2001), show that it cannot be analytically determined whether the first-best level of public input will exceed the second-best level. Consequently, a numerical approach has to be used, and under these circumstances RIM algorithm appears as the unique alternative which is able to solve the different scenarios considered in this example.

3.1 The model

We assume an economy of n identical households whose utility function is expressed by $u(x,l)$, where x is a private good used as numeraire and l the labor supply[14]. Let Y be the total endowment of time such that $h = Y - l$ is the leisure. Output in the economy is produced using labour services and a public input g according to the aggregate production function $F(nl,g)$. The type of returns to scale does not matter at the moment, and consequently using the Feehan's (1989) nomenclature, the public input can be treated as firm-augmenting (constant returns to scale in the private factor and the public input combined, creating rents) or as factor-augmenting (constant retuns to the private factor, and therefore scale economies in all inputs). Output can be costlessly used as x or g.

Labour market is perfectly competitive so that the wage rate ω is given by the marginal productivity of labour:

$$\omega = F_L(nl,g), \tag{6}$$

where firms take g as given. Profits may arise and defined as:

$$\pi = F(nl,g) - nl\omega, \tag{7}$$

[14]The properties of $u(x,l)$ are the standard ones to ensure a well-behaved function: strictly monotone, quasiconcave and twice differentiable.

which will be completely taxed away by government given their inelastic supply[15].

We distinguish two different tax settings. First, we consider a lump-sum tax T so that the representative household faces the following problem:

$$\begin{aligned} &Max \;\; u(x,l) \\ &s.t.: x = \omega l - T, \end{aligned} \tag{8}$$

which yields the labour supply $l(\omega, \omega Y - T)$ and the indirect utility function $V(\omega, \omega Y - T)$. It is to be assumed that $l_\omega \geq 0$.

The optimization problem of government in the first-best scenario is then as follows:

$$\begin{aligned} &\underset{R}{Max} \quad V(\omega(g), \omega Y - R) \\ &s.t.: g = nR, \end{aligned} \tag{9}$$

where $R = T + \pi(g, T)/n$ is the renevue per person[16].

A second scenario is that using a specific tax on labour τ. Under this tax setting, the consumer's optimization problem could be expressed as:

$$\begin{aligned} &Max \;\; u(x,l) \\ &s.t.: x = (\omega - \tau) l \end{aligned} \tag{10}$$

obtaining $l(\omega_N, \omega Y)$ and $V(\omega_N, \omega Y)$, where $\omega_N = \omega - \tau$ is the net wage rate. In this scenario, the optimization problem of government is given by:

$$\begin{aligned} &\underset{R}{Max} \quad V(\omega(g), \omega Y - TEB - R) \\ &s.t.: g = nR, \end{aligned} \tag{11}$$

with $R = \tau l + \pi(g, \tau)/n$ and TEB denoting the total excess burden.

3.2 Simulation and results

Next, we give an insight into the debate on the optimal level of public inputs using numerical procedures to solve particular cases. With this aim, we consider three different utility functions in an attempt to achieve results as general as possible and related to previous references on public goods.

[15] Pestieau (1976) analyzed how the optimal rule for the provision of public inputs has to be modified when these rents are not taxed away.

[16] It is useful here to consider that rents accrue to consumers before being taxing away by government.

Particularly, we have chosen the quasi-linear utility function (Gronberg and Liu, 2001); the Cobb-Douglas utility functions (Atkinson and Stern, 1974; Wilson, 1991a); and the CES utility function (Wilson, 1991b; Gaube, 2000). Specifically,

$$U(x,h) = x + 2h^{\frac{1}{2}} \tag{12}$$

$$U(x,h) = a\log x + (1-a)\log h \tag{13}$$

$$U(x,h) = (x^{\rho} + h^{\rho})^{\frac{1}{\rho}}\,, \tag{14}$$

where $Y = 24$, $a \in (0,1)$ and $\rho = 0.5$. The relevant point in our case comes from the specification of the production function because the different alternatives by defining how the private and public factors enter the production function have notable implications on the debate. In particular, whether this function exhibits constant returns to scale in public and private inputs (firm-augmenting public input) or only constant returns to the private factors (factor-augmenting public input) appear as key issues.

RIM algorithm has been used for solving all the scenarios summarized in Table 1. Additionally, the case of firm-augmenting public input has been solved using the NR algorithm. In such a way, the robustness of our proposal in this framework is checked. Equivalently, when a factor-augmenting public input is considered and problems of non-convexities problems can be avoided, we check whether the results coming from RIM method coincide with those obtained through the alternative approach by NM. With respect to this, one must be aware that RIM algorithm presents some advantages over NM, for instance the guarantee of obtaining the global optimum. The comparisons of RIM with NR and NM methods show that our approach takes advantages over both of them, specially when non-convexities arise[17].

Firm-augmenting public input

We assume a Cobb-Douglas production function given by $F(nl,g) = (nl)^{\alpha} g^{1-\alpha}$, where $\alpha \in (0,1)$. This specification creates firm-specific rents. As Pestieau (1976) proved, if these rents are also an argument in the consumer's indirect utility function, the optimal condition for the provision of public inputs is not the first-best one. However, recall that our model precisely establishes that all economic rents are taxed away by the government.

[17] The MATLAB routines used are available at
http://www.upo.es/econ/sanchez_fuentes/docs/research/RIMv2A.zip

Indeed, the controversy between the first-best and second-best level of public spending has no sense when the firm-augmenting public input creates rents which are completely taxed by the government. Under this scenario, the analytical solution of our model and its numerical resolution give the intuitive result that the optimal level of productive public spending must be exclusively financed with the economic rents.

For the simulation, we have taken $a \in \{0.1, \mathit{0.5}, 0.9\}$, $\alpha \in \{0.6, \mathit{0.7}, 0.8\}$ and $n \in \{1, \mathit{100}, 1000\}$ as the set of parameters to be used, where the benchmark values have been emphasized. The case of firm-augmenting public input is introduced with the aim of comparing the performance of RIM method under regular conditions with a widely used methodology, the standard NR algorithm.

Following the performance of RIM method, precision requirements and the bandwidth at different stages are set up (see Table 1). The definition of the vector E should take into consideration not only the aim of a high precision per se, but also the number of good values detected and others points which are close to the required precision but not satisfying them. In a sense, a trade-off between the precision and the number of compatible values appears.[18] The bandwidth is defined with the aim of obtaining a new decimal for the decision variable at each stage. With respect to the NR method, a standard implementation of this method has been used for solving the same problem in which the stop criterion equals to the maximum precision required for RIM algorithm.

INSERT TABLES 2-4 ABOUT HERE

Tables 2-4 compare the results achieved by using different methods. The coincidence of results is the main conclusion. As can be seen, a high coincidence of at least four decimals is obtained for each scenario and relevant variable. Despite of its higher computational costs, the robustness of these results support the suitable performance of RIM method in optimization problems with standard conditions.

Factor-augmenting public input

The main difference between the above case and this of factor-augmenting lies in the assumptions on the returns to scale in the production function.

[18] A different intuition of this issue is how the precision requirements can be relaxed to get good points within an area. RIM algorithm allows here to continue the searching process in "bad" areas decreasing conveniently the required precision.

Particularly, we assume again a Cobb-Douglas technology but exhibiting increasing returns in all the inputs (constant returns in labor): $F(nl,g) = nlg^{\beta}$, where $\beta \in (0,1)$. Under this framework, the debate on the level of public spending in alternative tax settings is reborn. Indeed, the use of lump-sum or distorting taxes are necessary as long as rents are null.[19] Here we have considered $a \in \{0.1, \mathit{0.5}, 0.9\}$, $\beta \in \{0.1, \mathit{0.2}, 0.3\}$ and $n \in \{1, \mathit{100}, 1000\}$ as the set of parameters to be taken account, where the benchmark values have been again emphasized.

Solving the government optimization problem with factor-augmenting public inputs is not as straightforward as before. Indeed, the NR algorithm presents some caveats when non-convex sets of constraints are involved. Note that this is our case because we have increasing returns in the production side of the model. Consequently, there is scope for a method such a RIM algorithm.

Recall that Table 1 described the parameters and method implemented under each scenario. In such a way, the specific properties of each one are used to solve the problem according to different criteria of selection between the points evaluated within each stage. First, RIM2 method follows the standard description of section 2. Second, RIM1u shows how RIM algorithm is adapted to solve an unconstrained optimization problem with one decision variable. The criterion followed consists of comparing consecutive values of the objective function with the aim of detecting changes in its monotonicity.[20] These changes could indicate the presence of local optima. Therefore, RIM method is adapted to obtain global optima within this framework, going beyond other direct-search methods which usually only ensure local optima.[21]

INSERT TABLES 5-7 ABOUT HERE

Tables 5-7 report the optimal levels of public inputs and other information in each scenario for each utility function. First-best levels are always higher than the second-best ones in line with the mainstream of previous literature on public goods, despite the feedback effect.[22] Several comments can be drawn regarding these tables. Firstly, the very different scale for the

[19] This fact justifies our exclusion of the cero from the initial sets, considering a minimum tax rate to be applied.

[20] As we are interested in maximizing our objective function, we will control for changes from increasing to decreasing areas.

[21] Variants of the original Nelder-Mead method are focused on avoiding problems related to this one. For further details, see Burmen *et al* (2005).

[22] The feedback effect could mitigate the distorsion caused by the tax according to some authors. For further information, see Martinez and Sanchez (2008).

tax rate obtained in each tax setting requires a previous knowledge for searching the optimal values. In other words, optimal values could be found in a more efficient way whether the search is done in the most suitable areas. Secondly, the coincidence of the results achieved with RIM1u and NM methods show the well performance of RIM algorithm for solving unconstrained optimization problems.

INSERT TABLES 8-9 ABOUT HERE

Tables 8 and 9 show the optimal path followed by RIM2 method in the different stages of the benchmark scenario for Quasi-linear and CES utility functions. The optimum achieved for each stage could be interpreted as the best choice according to the required level of precision. In addition, the values of elasticities indicate the percentage of change in the objective function caused by a deviation of 1 % from the optimal value of our decision variable. The decreasing optimal values obtained most of the times for the objective function in the different stages (V_{max}) come from the existence of a trade-off between the level of precision and the number of compatible values. The more precise results are demanded, the less points satisfy these requirements.

An additional advantage of RIM approach from standard methodologies, the detection of multiple optimal values, is observed within these tables. Mainly, we observe two different situations. On one hand, the same point may be evaluated twice whether it is separating two different intervals (see Table 8). This result can be explained as a redudant but unavoidable evaluation of our method. On the other hand, the same value of the objective function can be obtained using different values of the decision variables (Table 9). This result allows us to present how the additional information collected by RIM method during the resolution process is useful to decide which optimal value should be considered. Different criteria could be applied in our case to choose either the highest tax rate (to maximize the level of public input) or the lowest tax rate (with the aim of minimizing the damage caused by the tax), depending on the policy-maker preferences.

INSERT FIGURES 1-2 ABOUT HERE

Figures 1-2 show a complementary view of the searching process followed by RIM2 algorithm (recall that this approach only is implemented when nonconvexities appear) and already reported in Tables 8 and 9. They contain the good values found for each decision variable (marked by asterisks) and for some of the stages. Particularly, we show the first stage which allows to obtain a general perception of the initial set obtained and the three last stages, to observe how each stage continues the search on the 'good areas' found in the previous one. Alternatively, the 'bad' or exhausted areas are not taken into account. Again, the RIM method obtains more precise solutions as the level of precision increases and simultaneously avoids non-efficient computations. In addition, a crucial characteristic of RIM approach is exposed. According to the good areas detected in the first stage, non-convex sets of compatible points are detected. Under these conditions, standard methodologies fail out to solve these problems. By contrast, RIM method does not present any additional difficulty to solve them.

4 Concluding remarks

This paper has introduced a new numerical method, the Rational Iterative Multisection algorithm, which is able to obtain optimal values of optimization problems, even when they face some non-standard properties. In fact, we have dealt with two optimization problems: a problem with enough regularity properties and a non-convex problem as examples of its application. The method is based on a multisection iterative process of the initial set that evaluates the objetive function, obtaining compatible values of the decision variables under several precision requirements. The more stages are considered, the more precise values are obtained. Moreover, there exists a trade-off between the number of compatible values and the precision requirement imposed.

We have used a simple general equilibrium model with public inputs and two different tax settings: a lump-sum tax and a specific tax on labor. We have placed this exercise on the debate upon whether the first-best level of public inputs is higher than the second-best level. The government chooses the values of fiscal variables to maximize the utility of representative household. First, we have compared the new method to the well-known algorithm of Newton-Raphson, when the problem has enough regularity properties. This scenario refers to the case of firm-augmenting public inputs. The coincidence of the results is extremely high.

Second, an optimization problem where non-convex sets of constraints are involved has been also considered. This is the case of factor-augmenting public inputs. Under these conditions, RIM algorithm has relative advantages with respect to standard methods because the problems derived from multiple equilibria and corner solutions are avoided. In addition, an adaptation of RIM method for solving unconstrained optimization problems has been done which allows us to conclude that RIM algorithm goes beyond NM to ensure global optima in this framework. Our numerical results are clear: the level of public input in the first-best scenario always exceeds that of the second-best, in line with the mainstream of literature dealing with public inputs.

All in all, RIM approach becomes a useful tool for solving constrained optimization problems, in which relaxing the constraints is a relevant issue. An example of this could be a problem in which legal or constitutional arrangements imply that the government budget constraint has not to be fulfilled *strictu sensu*. Other applications of RIM algorithm could study the sensitivity of equilibrium values with respect to calibrated parameters in general equilibrium models. This goal can be carried out by RIM approach without the need of solving again the model, which will be necessary if other numerical approaches are used. Finally, RIM method could be implemented according as a paralell computing algorithm, with which the higher computational are mitigated.

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A Appendix

An intuitive explanation of the inter-stages procedure of RIM2 algorithm, supported by a graphical tool, is presented next. The figure below shows the transition from stage 1 to stage 2. In the first stage, the initial set of two decision variables, $\omega \times t$, has been discretized as a matrix 4×10.[23] All these points are candidates to be solutions of the optimization problem. Assume the points A, K, C, J and F are the points in which the restriction is fulfilled with the minimum precision E_1. Looking for more precise solutions, the areas $1, 2, 3, 4$ and 5 are built to form the initial set to be considered in the second stage.

INSERT FIGURE A1 ABOUT HERE

The way through which these areas 1-5 are formed is illustrated taking the coordinate ω_2 as a reference. The set of good values for the other decision variable t where the constraints are satisfied with a precision E_1 is $G(t_2, E_1) = \{t_3, t_4, t_7\}$. Grouping the consecutive subsequences of t, the minimum and maximum coordinates are, respectively: $\underline{h}_1^2 = 3$, $\bar{h}_1^2 = 4$, $\underline{h}_2^2 = \bar{h}_2^2 = 7$. Hence, the area to be used for the next stage coming from ω_2 will be $[\omega_1, \omega_3] \times [t_2, t_5] \cup [\omega_1, \omega_3] \times [t_6, t_8]$.

However, areas 1-5 are not achieved following *strictu sensu* the theoretical nomenclature explained above or considering directly the area obtained from any of the above points. By contrast, an "efficient" reorganization of the areas relative to each point is done. Obviously, the total areas are identical in both cases.

Next, the main features of this reorganization are described. We explain them by using some particular situations regarding the above figure.

- **The areas where different coordinates are involved should be integrated to optimize the procedure.** For instance, area 2 has been built on the basis of points A and J, which have different coordinates in t. Hence, our method does not duplicate some evaluations corresponding to the common area $[\omega_1, \omega_2] \times [t_7, t_8]$.

- **The good areas found for the interval of coordinates** $[\omega_{i-1}, \omega_{i+1}]$ **must be considered in separate intervals.** A good example of this is the situation of the areas 4 and 5. As long as this rule of reorganization had not taken place, the area $[\omega_3, \omega_4] \times [t_6, t_7]$ would

[23] For the sake of simplicity, the notation of decision variables is based on the nomenclature of section 3.

have been included in the second stage, and the objective function and the constraints would have been evaluated in this area, where it is unlikely to find a compatible value in the second stage.

- **The consecutive coordinates in t, which belong to the same coordinate in ω, are jointly considered in the definition of the area to be used in the next stage.** An illustration of this situation is given by areas 1 and 3. Points K and C belong to the same subsequence in t, with ω_2 as vertical coordinate.

At this point, areas 1-5 are used as the initial set in the second stage and RIM2 algorithm goes on searching for more precise solutions using the new grid of points subdividing these areas according to the parameters B and E.

Figures

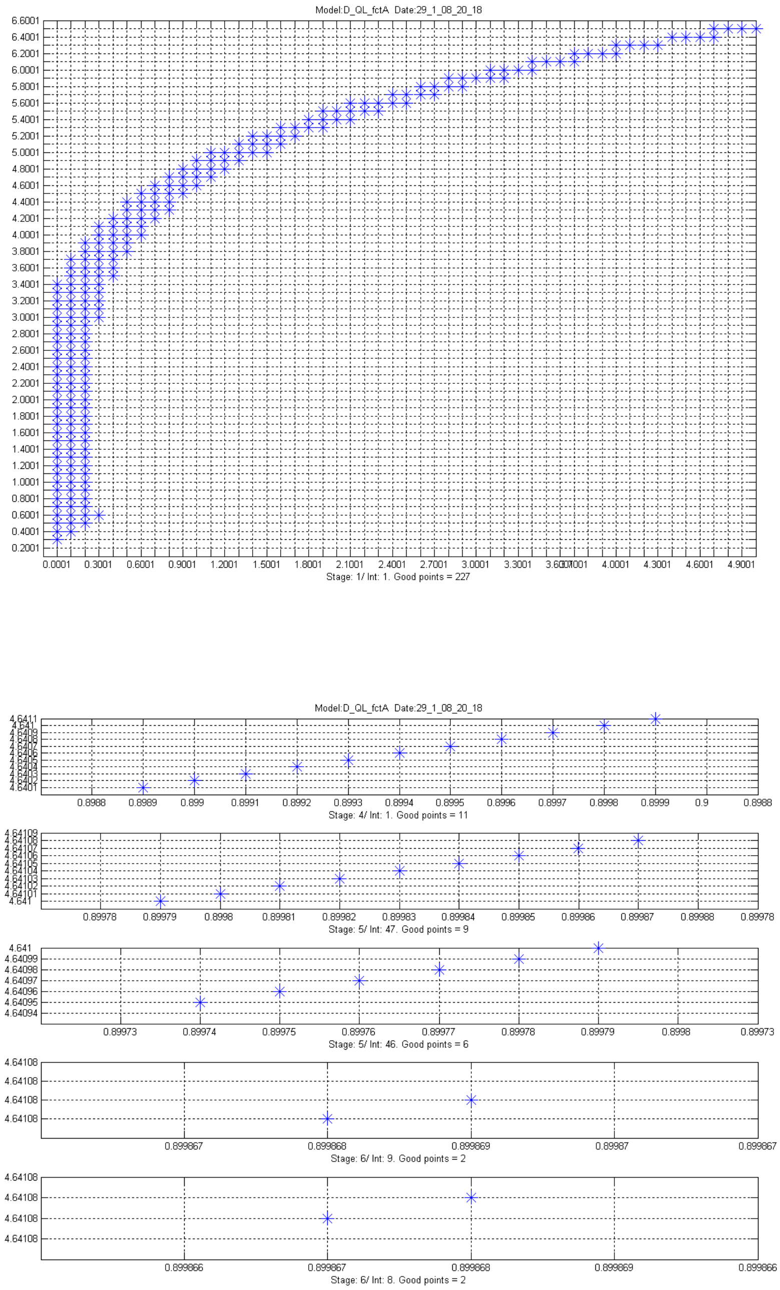

Figure 1: RIM2. Benchmark scenario. Good areas detected. Factor-augmenting public input. Quasi-linear utility. Distorting tax.

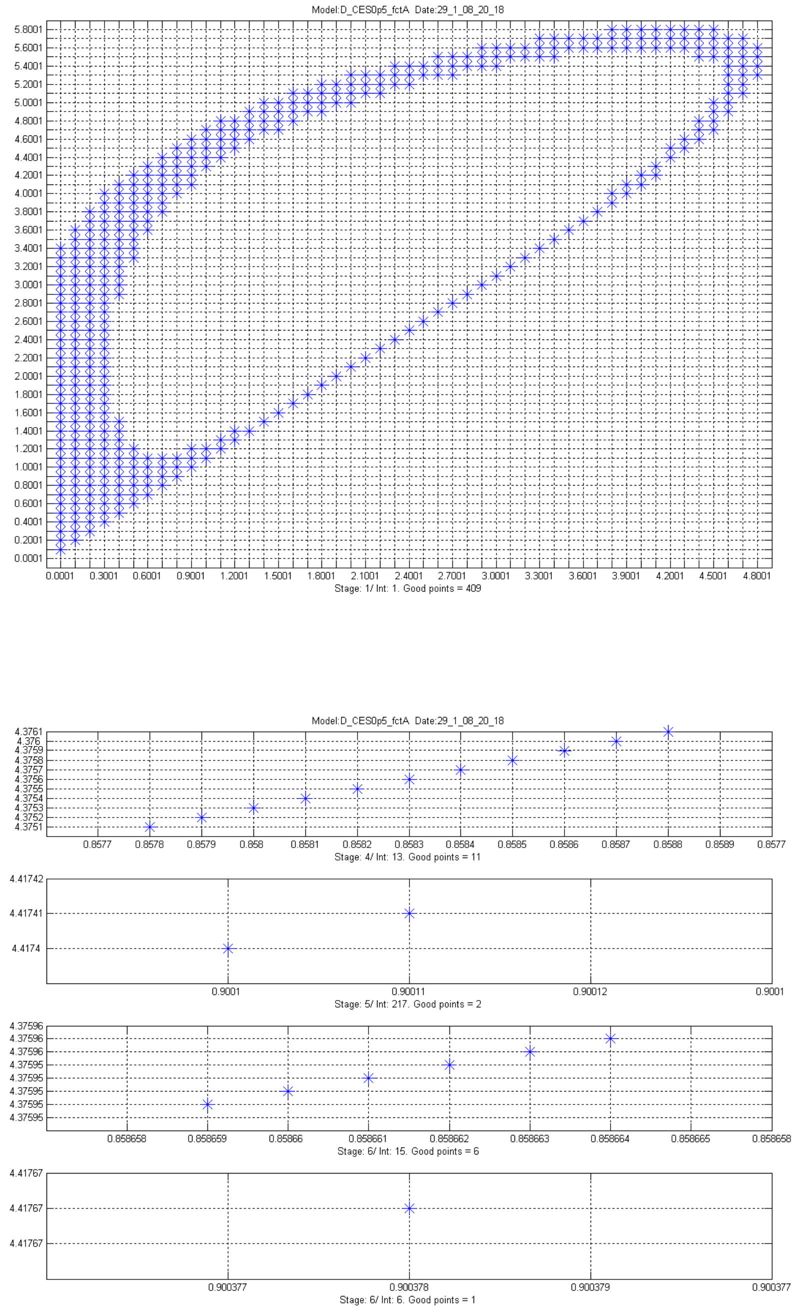

Figure 2: RIM2. Benchmark scenario. Good areas detected. Factor-augmenting public input. CES utility. Distorting tax.

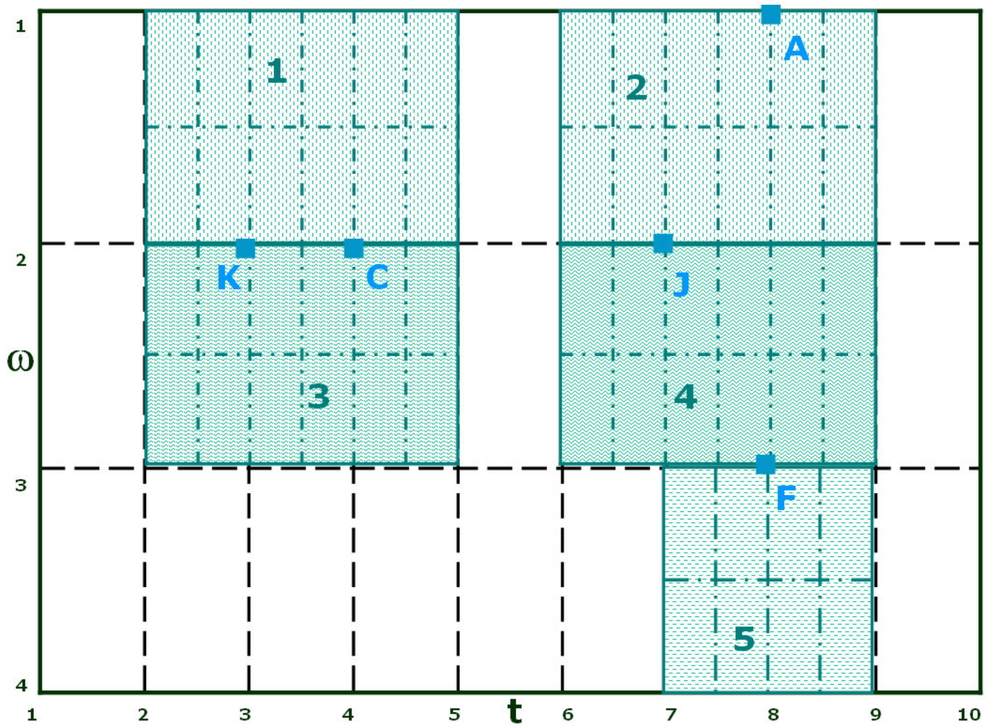

Figure A1: Inter-stages procedure.

Tables

Table 1: Simulation scenarios and methods

Scenarios	**Methods**	
	Firm-augmenting public input	
	RIM1	*Newton-Raphson*
LS $\rightarrow$ QL, CD and CES. D $\rightarrow$ QL, CD and CES.	$\omega \in [0.3, 0.6]$ $B = [10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}, 10^{-7}, 10^{-8}, 10^{-9}]$ $E = [40, 10, 1, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}]$	$\omega_0 = 0.5$ $E = 10^{-5}$
	Factor-augmenting public input	
	RIM1u	*Nelder-Mead*
LS $\rightarrow$ QL, CD and CES. D $\rightarrow$ CD.	$c \in [10^{-4}, 350 + 10^{-4}]$, $c = T, \tau$ $B = [1, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}, 10^{-7}]$ $E = [1, 1, 1, 1, 1, 1, 1, 1]$	$T_0^{QL} = T_0^{CES} = 10$ $T_0^{CD} = 0.5$ $E = 10^{-5}$
	RIM2	
D $\rightarrow$ QL and CES.	$(\omega, \tau) \in [10^{-4}, 20 + 10^{-4}] \times [10^{-4}, 5 + 10^{-4}]$ $B = [10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}]$ $E = [500, 50, 0.5, 0.05, 0.005, 10^{-4}]$	

Notes: 1) Tax settings: LS=Lump-sum, D=Distorting. 2) Utility functions: QL=Quasi-linear, CD=Cobb-Douglas, CES = CES utility function($\rho = 0.5$). 3) *RIMu* indicates that a different criterion for selection of points between stages has been used.

Table 2: Firm-augmenting. Quasi-linear utility.

	V_{max}	g	l	π
Benchmark $n = 100, \alpha = 0.7$	*12.4214* *(****)*	*327.2065* *(****)*	*18.2722* *(****)*	*327.2065* *(****)*
$\alpha = 0.6$	10.8876 (****)	316.4998 (****)	14.5749 (****)	316.4998 (****)
$\alpha = 0.8$	14.7090 (****)	274.2657 (****)	20.5061 (****)	274.2657 (****)
$n = 10$	12.4214 (****)	32.7207 (****)	18.2722 (****)	32.7207 (****)
$n = 1000$	12.4214 (****)	3272.0647 (****)	18.2722 (****)	3272.0647 (****)

Note: The number of asteriks (from 1 to 4 or more) indicate how many decimals are coincident between RIM and NR resolutions.

Table 3: Firm-augmenting. Cobb-Douglas utility.

	V_{max}	g	l	π
Benchmark $n = 100, \alpha = 0.7, a = 0.5$	*2.04858* (****)	*214.888* (****)	*12* (****)	*214.8877* (****)
$a = 0.1$	2.7657 (****)	42.9776 (****)	2.4 (****)	42.9776 (****)
$a = 0.9$	2.0676 (****)	386.7979 (****)	21.6 (****)	386.7979 (****)
$\alpha = 0.6$	1.9241 (****)	260.5841 (****)	12 (****)	260.5841 (****)
$\alpha = 0.8$	2.1722 (****)	160.4977 (****)	12 (****)	160.4977 (****)
$n = 10$	2.0486 (****)	21.4888 (****)	12 (****)	21.4888 (****)
$n = 1000$	2.0486 (****)	2148.8773 (****)	12 (****)	2148.8773 (****)

Note: The number of asteriks (from 1 to 4 or more) indicate how many decimals are coincident between RIM and NR resolutions

Table 4: Firm-augmenting. CES utility ($\rho = 0.5$).

	V_{max}	g	l	π
Benchmark $n = 100, \alpha = 0.7$	*34.0281* (****)	*126.6550* (****)	*7.0728* (****)	*126.6550* (****)
$\alpha = 0.6$	31.8175 (****)	128.0503 (****)	5.8968 (****)	128.0503 (****)
$\alpha = 0.8$	36.8398 (****)	111.8768 (****)	8.3647 (****)	111.8768 (****)
$n = 10$	34.0281 (****)	12.6655 (****)	7.0728 (****)	12.6655 (****)
$n = 1000$	34.0281 (****)	1266.5501 (****)	7.0728 (****)	1266.5501 (****)

Note: The number of asteriks (from 1 to 4 or more) indicate how many decimals are coincident between RIM and NR resolutions

Table 5: Factor-augmenting. Quasi-linear utility.

	$n = 100$	$n = 100$		$\beta = 0.2$	
	$\beta = 0.2$	$\beta = 0.1$	$\beta = 0.3$	$n = 10$	$n = 1000$
V_{max}: LS	90.0830	40.2563	281.8203	50.9174	159.9329
	(****)	(****)	(****)	(****)	(****)
D	90.0564	40.2491	281.8022	50.8935	159.2948
T, τ: LS	22.4139	4.3519	120.7290	12.5391	39.9232
	(****)	(****)	(****)	(****)	(****)
D	0.8999	0.1809	4.9566	0.5248	1.3531
g: LS	2241.3885	435.1883	12072.9012	125.3912	39923.1558
	(****)	(****)	(****)	(****)	(****)
D	2153.2517	427.4710	11914.3832	124.7701	32443.8427
l: LS	23.9543	23.7033	23.9964	23.8552	23.9856
	(****)	(****)	(****)	(****)	(****)
D	23.9286	23.6335	23.9927	23.7734	23.9773

Notes: 1) Benchmark scenario: $n = 100, \beta = 0.2$. LS=Lump-sum, D=Distorting. 2) The number of asteriks (from 1 to 4 or more) indicate how many decimals are coincident between RIM1u and NM resolutions .

Table 6: Factor-augmenting. Cobb-Douglas utility.

	$n = 100, a = 0.5$	$n = 100, \beta = 0.2$		$n = 100, a = 0.5$		$\beta = 0.2, a = 0.5$	
	$\beta = 0.2$	$a = 0.1$	$a = 0.9$	$\beta = 0.1$	$\beta = 0.3$	$n = 10$	$n = 1000$
V_{max}: LS	3.0654	2.9300	4.0200	2.6997	3.5869	2.7776	3.3532
	(****)	(****)	(****)	(****)	(****)	(****)	(****)
D	3.0584	2.9274	4.0175	2.6982	3.5679	2.7706	3.3462
	(****)	(****)	(****)	(****)	(****)	(****)	(****)
T, τ: LS	10.7761	1.6191	20.1986	2.1625	56.5836	6.0598	19.1629
	(****)	(****)	(****)	(****)	(****)	(****)	(****)
D	0.7872	0.5264	0.9118	0.1702	3.7383	0.4427	1.3999
	(****)	(****)	(****)	(****)	(****)	(****)	(****)
g: LS	1077.6083	161.9140	2019.8639	216.2484	5658.3633	60.5984	19162.8846
	(****)	(****)	(****)	(****)	(****)	(****)	(****)
D	944.6350	126.3431	1969.4941	204.2685	4486.0149	53.1207	16798.2513
	(****)	(****)	(****)	(****)	(****)	(****)	(****)
l: LS	13.3333	2.9268	22.0408	12.6316	14.1176	13.3333	13.3333
	(****)	(****)	(****)	(****)	(****)	(****)	(****)
D	12	2.4	21.6	12	12	12	12
	(****)	(****)	(****)	(****)	(****)	(****)	(****)

Notes: 1) Benchmark scenario: $n = 100, \beta = 0.2$ LS=Lump-sum, D=Distorting. 2) The number of asteriks (from 1 to 4 or more) indicate how many decimals are coincident between RIM1u and NM resolutions

Table 7: Factor-augmenting. CES utility ($\rho = 0.5$).

	$n = 100$	$n = 100$		$\beta = 0.2$	
	$\beta = 0.2$	$\beta = 0.1$	$\beta = 0.3$	$n = 10$	$n = 1000$
V_{max}: LS	*109.4471*	61.7688	298.7085	70.3698	179.2329
	(****)	(****)	(****)	(****)	(****)
D	*108.4150*	61.5920	295.7230	69.4549	178.1280
T, τ: LS	*18.3059*	2.7861	113.7856	8.8778	35.5307
	(****)	(****)	(****)	(****)	(****)
D	*0.90038*	0.17532	4.84211	0.47327	1.57368
g: LS	*1830.5908*	278.6116	11378.5635	88.7784	35530.7014
	(****)	(****)	(****)	(****)	(****)
D	*1682.5437*	256.8082	10677.9327	74.3357	32679.6423
l: LS	*20.3724*	15.8672	23.0218	18.0974	21.8501
	(****)	(****)	(****)	(****)	(****)
D	*18.6871*	14.6481	22.0522	15.7007	20.7664

Notes: 1) Benchmark scenario: $n = 100, \beta = 0.2$. LS=Lump-sum, D=Distorting. 2) The number of asteriks (from 1 to 4 or more) indicate how many decimals are coincident between RIM1u and NM resolutions.

Table 8: Optimal path of RIM2 in different stages. Factor-augmenting public input. Quasi-linear utility. Distorting tax setting.

Stage	Interval	ω	τ	$V_{\max}$	R	E	e^V_ω	e^V_τ
1	1	4.40010	0.50010	93.85641	452.3976	500	1.12206	-0.12753
2	45	4.63010	0.87010	90.50596	45.81765	50	1.22417	-0.23005
3	54	4.63010	0.88910	90.05131	0.41608	0.5	1.23032	-0.23626
3	53	4.63010	0.88910	90.05131	0.41608	0.5	1.23031	-0.23625
4	1	4.63030	0.88910	90.05609	0.02895	0.05	1.23296	-0.23890
5	47	4.64104	0.89983	90.05633	0.00184	0.005	1.23316	-0.23909
5	46	4.64097	0.89976	90.05633	0.00328	0.005	1.23314	-0.23907
6	9	4.64108	0.89987	90.05638	$9 * 10^{-6}$	0.0001	1.23316	-0.23910
6	8	4.64108	0.89987	90.05638	0.00008	0.0001	1.23316	-0.23910

Note: Benchmark scenario: $n = 100, \beta = 0.2$. R is the precision with which the constraints are satisfied. e^V_X is the elasticity of the objective function V with respect to $X = \omega, \tau$.

Table 9: Optimal path of RIM2 in different stages. Factor-augmenting public input. CES utility ($\rho = 0.5$). Distorting tax setting.

Stage	Interval	ω	τ	$V_{\max}$	R	E	e_{ω}^{V}	e_{τ}^{V}
1	1	4.00010	0.30010	112.8000	457.1306	500	0.8511	-0.0639
2	77	4.40010	0.86010	108.9600	39.7881	50	0.9692	-0.1894
2	75	4.33010	0.79010	108.9600	43.6968	50	0.9538	-0.1740
2	73	4.29010	0.75010	108.9600	49.5277	50	0.9450	-0.1652
3	48	4.37110	0.85410	108.4080	0.3253	0.5	0.9677	-0.1891
3	47	4.36110	0.84410	108.4080	0.1918	0.5	0.9655	-0.1869
3	46	4.35810	0.84110	108.4080	0.3793	0.5	0.9648	-0.1862
3	34	4.43010	0.91310	108.4080	0.0620	0.5	0.9808	-0.2021
3	33	4.42510	0.90810	108.4080	0.2022	0.5	0.9797	-0.2010
4	13	4.37560	0.85830	108.4152	0.0267	0.05	0.9686	-0.1900
5	217	4.41740	0.90010	108.4152	0.0045	0.005	0.9779	-0.1993
6	15	4.37595	0.85866	108.4150	0.00009	0.0001	0.9687	-0.1900
6	6	4.41767	0.90038	108.4150	0.00008	0.0001	0.9779	-0.1993

Note: Benchmark scenario: $n = 100, \beta = 0.2$. R is the precision with which the constraints are satisfied. e_X^V is the elasticity of the objective function V with respect to $X = \omega, \tau$.